Constructive approximation and special functions: theory and applications

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Special Functions (SF) appear in (almost) all context of Mathematics and other Sciences.

As Alberto Grunbaum one time said: “Special functions are to mathematics what pipes are to a house: nobody wants to exhibit them openly but nothing works without them”.

Key fact: Almost all SF of Mathematical-Physics are solutions of
\[ \sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0 \]
with
\[
\begin{cases}
\deg \sigma \leq 2 \\
\deg \tau \leq 1 \\
\lambda = \text{const}
\end{cases}
\]
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In this course we will show a little piece of such universe.

1. Historical introduction to OPs & its relation with QM
2. The classical OPs: properties and characterization theorems
3. Some general properties of OPs: TTRR, zeros, quadrature formulas, ...
The first OP family: The Legendre polynomials

The 1st family of OP appears in the XVIII century: *Sur l’attraction des sphéroïdes* (1785)

\[
P(r, \theta, 0) = \int \int \int_{\Omega} \frac{(r - r') \cos \gamma}{(r^2 - 2rr' \cos \gamma + r'^2)^{\frac{3}{2}}} r'^2 \sin \theta' d\theta' d\phi' dr'
\]

\[
\clubsuit = \frac{1}{r^2} \left\{ 1 + 3P_2(\cos \gamma) \frac{r'^2}{r^2} + 5P_4(\cos \gamma) \frac{r'^4}{r^4} + \ldots \right\}
\]

In a 2° paper published in 1787 he proves the 1st orthogonality relation

\[
\int_0^1 P_{2n}(x) P_{2m}(x) dx = \frac{1}{4m + 1} \delta_{n,m}.
\]

Latter on he stablished the general orthogonality relation

\[
\int_0^1 P_n(x) P_m(x) dx = \frac{2}{2n + 1} \delta_{n,m}
\]

Adrien–Marie Legendre
The first OP family: The Legendre polynomials

He notices that the zeros of $P_n$ have a very nice properties: were real, simple, and lies in $(0, 1)$

He proves that “any” $f(x^2)$ can be expanded as $f(x^2) = \sum_{n=0}^{\infty} c_n P_{2n}(x), \; \exists! c_n$

Also he establish that the $P_n$ satisfy the differential equation

$$(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n + 1)P_n(x) = 0$$

Olinde Rodrigues (1816): Rodrigues formula $P_n(x) = \frac{1}{2^n n!} \frac{d^n(x^2 - 1)^n}{dx^n}$
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The 1º OP family appears in the context of celestial mechanics!
Historical introduction and Motivation

Solving the Schrödinger equation: The Nikiforov-Uvarov method

Laguerre polynomials & Approximation theory

Abel (1828) y Lagrange (1810) – Chebyshev (1859)
& Posse (1873) – Sojotkin (1873) – Laguerre (1879)

\[
\int_0^\infty L_n(x)L_m(x)x^\alpha e^{-x}dx = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{n,m}.
\]

SODE \quad x(L_n^\alpha)^{'''} + (\alpha + 1 - x)(L_n^\alpha)' + nL_n^\alpha = 0

Sonin (1880) considered the case \( \alpha > -1 \).

What was the problem behind this family of OPs?

The relations of the integral \( \int_x^\infty \frac{e^{-x}}{x} dx \) and the continuous fraction

\[
\int_x^\infty \frac{e^{-x}}{x} dx = \frac{e^{-x}}{x + 1 - \frac{1}{x + 3 - \ldots}},
\]

\[
\approx e^{-x} \frac{\phi_m(x)}{L_m(x)},
\]
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Hermite polynomials & series expansions

Laplace (1810) – Chebyshev (1859) – **Hermite** (1864)

In *Sur un nouveau développement en série des fonctions* C.R.Acad. Sci. Paris, Hermite consider the expansion

\[ F(x) = A_0 H_0 + A_1 H_1 + \cdots A_n H_n + \cdots , \]

where \( \frac{d^n e^{-x^2}}{dx^n} = e^{-x^2} H_n(x) \) (multivariable case).

\( H_n \) was defined using a Rodrigues-type formula!

**SODE**

\[ H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \]

**TTRR**

\[ H_{n+1} + 2xH_n(x) + 2nH_{n-1}(x) = 0 \]

and the **orthogonality** relation

\[ \int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2} \, dx = 2^n n! \sqrt{\pi} \delta_{n,m}. \]
Historical introduction and Motivation

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The hypergeometric function \( _2F_1 \)

L. Euler & C. Gauss

Euler in *Instituciones Calculi Integralis* (1769) introduced the SODE

\[
x(1 - x)y'' + [\gamma - (\alpha + \beta + 1)x]y' - \alpha\beta y = 0
\]

and obtain its solution, now denoted by \( _2F_1 \),

\[
_2F_1\left( \alpha, \beta; \gamma \bigg| x \right) = 1 + \frac{\alpha\beta}{1\gamma}x + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{2\gamma(\gamma + 1)}x^2 + \cdots .
\]

Gauss in *Disquisitiones generales circa seriem infinitam* ..., (1876) did a complete study of the SODE and the series \( _2F_1 \). In particular he establish that almost all functions can be written in terms of \( _2F_1 \)

\[
(1 + z)^a = _2F_1(-a, b; b|-z), \quad \log(1 + z) = _2F_1(1, 1; 2|-z), \quad J_n(z) = \cdots
\]
Using the $2F_1$ Jacobi, in a paper published in 1859, define a new family of OP:

$$P_{n}^{\alpha,\beta}(x) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)n!} \frac{\Gamma(n + \alpha + \beta + 1)}{\alpha + 1} \left(\begin{array}{c}
-n, n + \alpha + \beta + 1 \\
\alpha + 1 \end{array}\right)_{2F_1} \left(\frac{1 - x}{2}\right)$$

which generalizes the Legendre ones $P_n = P_{n}^{0,0}$. Among other properties he obtained the orthogonality relation and the SODE

$$\int_{-1}^{1} P_{n}^{\alpha,\beta}(x)P_{m}^{\alpha,\beta}(x)(1-x)^{\alpha}(1+x)^{\beta} \, dx = \frac{2^{\alpha+\beta+1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1)\Gamma(n + \alpha + \beta + 1)n!} \delta_{n,m}$$

and the SODE

$$(1 - x^2)(P_{n}^{\alpha,\beta})'' + (\beta - \alpha - (\alpha + \beta + 2)x)(P_{n}^{\alpha,\beta})' + n(n + \alpha + \beta + 1)P_{n}^{\alpha,\beta} = 0$$
The general theory: From Stieljester and Chebyshev to Szegö

\[ x - b_0 - \frac{a_0^2}{\cdot} \]
\[ x - b_1 - \frac{a_1^2}{\cdot} \]
\[ x - b_2 - \frac{a_2^2}{\cdot} \]
\[ x - b_3 - \cdot \]

Thomas Stieltjes Jr.
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The general theory: From Stieljests and Chebyshhev to Szegö

\[
\begin{align*}
\frac{a_0^2}{x - b_0} & \quad \frac{a_1^2}{x - b_1} \\
\frac{a_2^2}{x - b_2} & \quad \frac{a_3^2}{x - b_3} \\
& \quad \ddots 
\end{align*}
\]

Thomas Stieltjes Jr.

If \( a_k = 0 \), \( \forall k \geq n + 1 \) \( \Rightarrow \) \( f_n(x) = \frac{1}{a_1} \frac{p_{n-1}^{(1)}(x)}{p_n(x)} \)

being \( p_n(x) \) and \( p_{n-1}^{(1)}(x) \) the solution of the

\[\text{TTRR} \quad x r_n(x) = a_{n+1} r_{n+1}(x) + b_n r_n(x) + a_n r_{n-1}(x), \quad n \geq 0, \]

where \( r_{-1}(x) = 0, r_0(x) = 1 \) and \( r_{-1}(x) = 1, r_0(x) = 0 \), resp.

\textbf{Theorem:}\ If \( (p_n)_n \) satisfies the TTRR, \( a_{k+1} > 0, b_k \in \mathbb{R} \) &

\( p_{-1}(x) = 0, p_0(x) = 1 \), \( \Rightarrow \) \( \exists \) measure s.t.

\[\int_{-\infty}^{\infty} p_n(x) p_m(x) d\mu(x) = \delta_{n,m}.\]
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Chebyshev considered several problems that lead to OPs:
E.g. the study of the continued fractions of \[ \int_a^b \frac{\rho(x)dx}{z-x}. \]

Pafnuti Chebyshev
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In *Théorie des mécanismes ...* (1854)

Chebyshev polynomials of the 1\(^{\circ}\) kind \( T_n(x) = P_n^{\frac{-1}{2},-\frac{1}{2}} \).

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p_n(x) = \frac{\cos(n \arccos x)}{2^{n-1}}, \quad x \in [-1, 1].
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In *Sur l'interpolation des valeurs équidistantes* (1875)

he considered the following problem:

Given \( x_0, \ldots, x_N \) & \( y_0, \ldots, y_N \), to find the \( A_m \) in

\[
y \approx A_0 P_0(x) + \cdots + A_k P_k(x), \quad (k < N) \quad \text{s.t.}
\]

\[
\sum_{i=0}^{N-1} \rho(x_i)[y_i - A_0 P_0(x_i) - \cdots - A_k P_k(x_i)]^2 = \min., \quad x_{i+1} = x_i + i, \quad \Rightarrow
\]

Chebyshev polynomials of discrete variables ...
The general theory of OP was developed at the end of the XIX century and beginning of XX. This process finished with the appearance of his classical book *Orthogonal polynomials* (1939, AMS).

**Definition:** Given the sequence \((P_n)_n\), \(\deg P_n = n\), we say that \((P_n)_n\) is an OPS w.r.t. measure \(\mu\) if

\[
\int_{\mathbb{R}} P_n(x)P_m(x)d\mu(x) = \delta_{n,m}d_n, \quad n, m = 0, 1, 2, \ldots
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Historical introduction and Motivation

Solving the Schrödinger equation: The Nikiforov-Uvarov method

Gabor Szegő and the general theory of OP

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\int_{\mathbb{R}} P_n(x)P_m(x)d\mu(x) = \delta_{n,m}d_n, \quad n, m = 0, 1, 2, \ldots \quad d\mu(x) = \rho(x)dx, \quad d_n > 0.
\]
Before establish the link between OP & QM we should say something about Physical Theories and, in particular, on QM.

In general, any Physical Theory has three main ingredients:

1. **Formalism**: A set of symbols and rules of deductions. With the symbols one can build statements and propositions. The rules of deductions allow us to deduce new propositions from those already given.

2. **The dynamical laws**: Some relations that some of the basic objects of the formalism must satisfy and that allow us to *predict the future*.

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In QM the formalism are the vectors and self-adjoint operators in a **Hilbert space**, the dynamical law in QM is the **Schrödinger Equation**, and the Theory of measurements gives the correspondence rules.
Among all physical systems there are two that paradigmatic:

- The quantum harmonic oscillator, whose stationary Schrödinger Eq. is

\[ \psi''(\xi) + (\varepsilon - \xi^2) \psi(\xi) = 0, \quad \psi(\xi) \text{ wave function} \]

- The Hydrogen atom

\[
\begin{bmatrix}
\Delta_r + \frac{1}{r^2} \Delta_\zeta \\
\end{bmatrix} \psi(r, \theta, \phi) + [\varepsilon - v(r)] \psi(r, \theta, \phi) = 0.
\]

Using separation of variables the radial part becomes

\[ R''(\zeta) + \left[ 2 \left( \varepsilon + \frac{1}{\zeta} \right) - \frac{l(l+1)}{\zeta^2} \right] R(\zeta) = 0. \]

In both cases it looks like

\[ u''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)} u'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)} u(z) = 0. \]
The generalized hypergeometric equation

**Definition:** The generalized hypergeometric equation (GHQ) is the equation

\[ u''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)} u'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)} u(z) = 0, \]

where \( \tilde{\tau}(z) \) is a polynomial of degree at most 1 and \( \sigma(z) \) & \( \tilde{\sigma}(z) \) are polynomials of degree at most 2.
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Let us make the change \( u(z) = \phi(z)y(z) \),

\[ y'''(z) + \left( 2 \frac{\phi'(z)}{\phi(z)} + \frac{\tilde{\tau}(z)}{\sigma(z)} \right) y'(z) + \left( \frac{\phi''(z)}{\phi(z)} + \frac{\phi'(z)\tilde{\tau}(z)}{\phi(z)\sigma(z)} + \frac{\tilde{\sigma}(z)}{\sigma^2(z)} \right) y(z) = 0. \]

**Idea?**
The generalized hypergeometric equation

The idea is to transform the GHE into a more simple –or simpler– Eq.

\[ u''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)} u'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)} u(z) = 0 \]

After changes GHE becomes into

\[ y''(z) + \left( 2 \frac{\phi'(z)}{\phi(z)} + \frac{\tilde{\tau}(z)}{\sigma(z)} \right) y'(z) + \left( \frac{\phi''(z)}{\phi(z)} + \frac{\phi'(z)\tilde{\tau}(z)}{\phi(z)\sigma(z)} + \frac{\tilde{\sigma}(z)}{\sigma^2(z)} \right) y(z) = 0. \]
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In order that Eq. and Eq. are, at least, of the same type we should have 1)

\[ 2 \frac{\phi'(z)}{\phi(z)} + \frac{\tilde{\tau}(z)}{\sigma(z)} = \frac{\tau(z)}{\sigma(z)}, \quad \frac{\phi'(z)}{\phi(z)} = \frac{\tau(z) - \tilde{\tau}(z)}{2\sigma(z)} = \frac{\pi(z)}{\sigma(z)}, \]

where \( \tau \) is a polynomial of degree at most 1, and therefore also \( \pi \).
The generalized hypergeometric equation

I.e. the original Eq.

\[ u''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)} u'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)} u(z) = 0 \]

transforms into the following equivalent one

\[ y''(z) + \frac{\tau(z)}{\sigma(z)} y'(z) + \frac{\overline{\sigma}(z)}{\sigma^2(z)} y(z) = 0, \]

\[ \tau(z) = \tilde{\tau}(z) + 2\pi(z), \]

\[ \overline{\sigma}(z) = \tilde{\sigma}(z) + \pi^2(z) + \pi(z)[\tilde{\tau}(z) - \sigma'(z)] + \pi'(z)\sigma(z). \]
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So what? This solve our problem?
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So what? This solve our problem? **YES!**
The generalized hypergeometric equation

\[ y''(z) + \frac{\tau(z)}{\sigma(z)} y'(z) + \frac{\bar{\sigma}(z)}{\sigma^2(z)} y(z) = 0 \]

\( \bar{\sigma} \) is (at most) 2 degree polynomial, then we will force him to be proportional to \( \sigma \), i.e.

\[ \bar{\sigma}(z) = \lambda \sigma(z) \]
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The generalized hypergeometric equation

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This is possible since \(\bar{\sigma}(z) = \bar{\sigma}(z) + \pi^2(z) + \pi(z)[\bar{\tau}(z) - \sigma'(z)] + \pi'(z)\sigma(z)\) has 2 indeterminate coefficients – those of \(\pi\) – and \(\lambda\) is an unknown. Then we have three equations and three unknowns.
The generalized hypergeometric equation

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Therefore, our starting Eq. becomes into

**Definition:** The hypergeometric equation

\[ \sigma(z) y'' + \tau(z) y' + \lambda y = 0. \]
Let us show how to compute the unknowns $\pi$ and $\lambda$. Since $\tilde{\sigma} = \lambda \sigma(z)$, then

$$\tilde{\sigma}(z) + \pi^2(z) + \pi(z)[\tilde{\tau}(z) - \sigma'(z)] + \pi'(z)\sigma(z) = \lambda \sigma(z),$$

or, equivalently,

$$\pi^2(z) + [\tilde{\tau}(z) - \sigma'(z)]\pi(z) + \{\tilde{\sigma}(z) - [\lambda - \pi'(z)]\sigma(z)\} = 0.$$
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Suposse that $k = \lambda - \pi'(z)$ is known. Then we obtain a second order equation for $\pi(z)$. 
The generalized hypergeometric equation

\[ \pi^2(z) + [\tilde{\tau}(z) - \sigma(z)]\pi(z) + \{\tilde{\sigma}(z) - [\lambda - \pi'(z)]\sigma(z)\} = 0 \quad \Rightarrow \]

\[ \pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)}. \]
The generalized hypergeometric equation

\[ \pi^2(z) + [\tilde{\tau}(z) - \sigma(z)]\pi(z) + \{\tilde{\sigma}(z) - [\lambda - \pi'(z)]\sigma(z)\} = 0 \implies \]

\[ \pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)}. \]

But the polynomial \( \pi(z) \) is at most of degree 1, thus the polynomial \( \Upsilon(z) \)

\[ \Upsilon(z) = \left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z) \]

is necessarily a perfect square \( (\Upsilon(z) = \eta^2(z), \deg \eta \leq 1) \).
The generalized hypergeometric equation

\[
\pi^2(z) + [\tilde{\tau}(z) - \sigma(z)] \pi(z) + \{\tilde{\sigma}(z) - [\lambda - \pi'(z)] \sigma(z)\} = 0 \quad \Rightarrow
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\Upsilon(z) = \left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k \sigma(z)
\]

is necessarily a perfect square (\(\Upsilon(z) = \eta^2(z)\), \(\deg \eta \leq 1\)). Therefore, its discriminant should be zero. This leads to a condition for obtaining \(k\). As far as we find \(k\) we substitute it in the formula for \(\pi\). Then, knowing \(\pi\) we use \(k = \lambda - \pi'(z)\) to obtain \(\lambda\).
The generalized hypergeometric equation

\[ \pi^2(z) + [\tilde{\tau}(z) - \sigma(z)]\pi(z) + \{\tilde{\sigma}(z) - [\lambda - \pi'(z)]\sigma(z)\} = 0 \Rightarrow \]

\[ \pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)}. \]

But the polynomial \( \pi(z) \) is at most of degree 1, thus the polynomial \( \Upsilon(z) \)

\[ \Upsilon(z) = \left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z) \]

is necessarily a perfect square (\( \Upsilon(z) = \eta^2(z), \deg \eta \leq 1 \)). Therefore, its discriminant should be zero. This leads to a condition for obtaining \( k \). As far as we find \( k \) we substitute it in the formula for \( \pi \). Then, knowing \( \pi \) we use \( k = \lambda - \pi'(z) \) to obtain \( \lambda \).

Therefore, the analysis of some Phys.Math. problems leads to the hypergeometric equation \( \sigma(z)y''' + \tau(z)y' + \lambda y = 0 \).
Constructive approximation and special functions: theory and applications

Renato Álvarez-Nodarse

Sevilla, March 2010
Definition: measures and linear functionals

Let us consider the Hilbert space $L^2_\mu[a, b]$ with the scalar product

$$\langle f, g \rangle = \int_a^b f(x)g(x) d\mu(x).$$

In this space two vectors (functions) are orthogonal ($f \perp g$) if $\langle f, g \rangle = 0$.

The above inner product can be written in terms of linear functionals:

$$\mathcal{L} : L_\mu(a, b) \mapsto \mathbb{C}, \quad \mathcal{L}[f] = \int_a^b f(x) d\mu(x), \quad f \perp g \iff \mathcal{L}[f \cdot g] = 0.$$

In the following we will use the orthogonality not w.r.t. an inner product, but to a linear functional.
Definition: measures and linear functionals

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The above inner product can be written in terms of linear functionals:

$$L : L_{\mu}(a, b) \mapsto \mathbb{C}, \quad L[f] = \int_{a}^{b} f(x)d\mu(x), \quad f \perp g \iff L[f \cdot g] = 0.$$

In the following we will use the orthogonality not w.r.t. an inner product, but to a linear functional.

Definition: We say that $L$ is a moment functional defined by the sequence $(\mu_n)_n$ of moments of order $n$, $\mu_n \in \mathbb{C}$, if $L$ is linear on $\mathbb{P}$ and

$$L[x^n] = \mu_n, \quad n = 0, 1, 2, \ldots.$$
Definition: Given a sequence of polynomials \((P_n)_n\), we say that \((P_n)_n\) is an orthogonal polynomials sequence (OPS) w.r.t. \(L\) if:

1. \(\deg P_n = n, \forall n = 0, 1, 2, \ldots\),
2. \(L[P_nP_m] = 0, \quad m \neq n, \forall n, m = 0, 1, 2, \ldots\),
3. \(L[P_n^2] \neq 0, \quad \forall n = 0, 1, 2, \ldots\).

If \(\forall n, \ P_n(x) = x^n + b_n x^{n-1} + \cdots\), we say that the SOP is monic (MSOP).

If \(\forall n, \ L[P_n^2] = 1\), we say that the SOP is orthonormal.
Definition: Given a sequence of polynomials \((P_n)_n\), we say that \((P_n)_n\) is an orthogonal polynomials sequence (OPS) w.r.t. \(\mathcal{L}\) if:

1. \(\deg P_n = n, \forall n = 0, 1, 2, \ldots,\)
2. \(\mathcal{L}[P_nP_m] = 0, \quad m \neq n, \forall n, m = 0, 1, 2, \ldots,\)
3. \(\mathcal{L}[P_n^2] \neq 0, \forall n = 0, 1, 2, \ldots.\)

If \(\forall n, P_n(x) = x^n + b_n x^{n-1} + \cdots\), we say that the SOP is monic (MSOP).

If \(\forall n, \mathcal{L}[P_n^2] = 1\), we say that the SOP is orthonormal.

Theorem: Let \(\mathcal{L}\) a moment functional and \((P_n)_n\) a polynomial sequence s.t. \(\deg P_n = n\). Then the following statements are equivalents:

1. \((P_n)_n\) is an OPS w.r.t. \(\mathcal{L}\).
2. \(\mathcal{L}[\pi P_n] = 0, \forall \pi \in \mathbb{P}, \deg \pi = m < n, \) and \(\mathcal{L}[\pi P_n] \neq 0, \) if \(\deg \pi = n\).
3. \(\mathcal{L}[x^m P_n(x)] = K_n \delta_{nm}, \) where \(K_n \neq 0, m = 0, 1, \ldots, n.\)
Let \((P_n)_n\) an OPS w.r.t \(L\). Then, \(\forall \pi, \deg \pi = n\) one has
\[
\pi(x) = \sum_{k=0}^{n} c_k P_k(x), \quad \text{where} \quad c_k = \frac{L[\pi P_k]}{L[P_k^2]}, \quad k = 0, 1, \ldots, n.
\]

Moreover, every polynomial \(P_n\) of the OPS is uniquely determined up to a multiplicative factor.

Since \(\pi(x) = \sum_{j=0}^{n} c_j P_k(x)\), multiplying by \(P_k\) and applying \(L\) we obtain \(c_k\).

Suppose now that there exists a polynomial \(Q_n \perp P_0, \ldots, P_{n-1}\), \(\deg Q_n = n\) such that \(Q_n(x) \neq \alpha_n P_n(x)\), being \(\alpha_n \in \mathbb{C}\). Then expanding \(Q_n\) in the basis \((P_n)_n\) we obtain that \(c_k = L[P_k Q_n] = 0\), \(\forall k < n\), i.e. \(Q_n(x) = c_n P_n(x)\), which is a contradiction.
The general theory of orthogonal polynomials

Unicity of OPS

Let \((P_n)_n\) an OPS w.r.t \(\mathcal{L}\). Then, \(\forall \pi, \deg \pi = n\) one has

\[\pi(x) = \sum_{k=0}^{n} c_k P_k(x), \quad \text{where} \quad c_k = \frac{\mathcal{L}[\pi P_k]}{\mathcal{L}[P_k^2]}, \quad k = 0, 1, \ldots, n.\]

Moreover, every polynomial \(P_n\) of the OPS is uniquely determined up to a multiplicative factor.

Since \(\pi(x) = \sum_{j=0}^{n} c_j P_k(x)\), multiplying by \(P_k\) and applying \(\mathcal{L}\) we obtain \(c_k\).

Suppose now that there exists a polynomial \(Q_n \perp P_0, \ldots, P_{n-1}\), \(\deg Q_n = n\) such that \(Q_n(x) \neq \alpha_n P_n(x)\), being \(\alpha_n \in \mathbb{C}\). Then expanding \(Q_n\) in the basis \((P_n)_n\) we obtain that \(c_k = \mathcal{L}[P_k Q_n] = 0\), \(\forall k < n\), i.e. \(Q_n(x) = c_n P_n(x)\), which is a contradiction.

In the following by \(\Delta_n\) we will denote the determinat

\[
\Delta_n = \begin{vmatrix}
\mu_0 & \mu_1 & \cdots & \mu_n \\
\mu_1 & \mu_2 & \cdots & \mu_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_n & \mu_{n+1} & \cdots & \mu_{2n}
\end{vmatrix}.
\]
Theorem: Let $\mathcal{L}$ be the functional associated to the sequence of moments $(\mu_n)_n$ and let $(P_n)_n$ a polynomial sequence.

Then $(P_n)_n$ is an OPS w.r.t. $\mathcal{L}$ iff $\Delta_n \neq 0$, $\forall n$. Moreover the leading coefficient $a_n$ in $P_n(x) = a_n x^n + \cdots$ is given by $a_n = \frac{K_n \Delta_{n-1}}{\Delta_n}$.
Theorem: Let $\mathcal{L}$ be the functional associated to the sequence of moments $(\mu_n)_n$ and let $(P_n)_n$ a polynomial sequence. Then $(P_n)_n$ is an OPS w.r.t. $\mathcal{L}$ iff $\Delta_n \neq 0$, $\forall n$. Moreover the leading coefficient $a_n$ in $P_n(x) = a_n x^n + \cdots$ is given by $a_n = \frac{K_n \Delta_{n-1}}{\Delta_n}$.

Remark: Not any sequence $(\mu_n)_n$ defines an OPS. An example is the sequences $(\mu_k)_k$ s.t. $\mu_0 = 0$, or $\mu_0 = \mu_1 = \mu_2$ (why?)

If $\mathcal{L}$ has an integral representation $\mathcal{L}[f] = \int_a^b f(x) \rho(x) dx$, $\rho(x) > 0$, then it can be proved that there exists the corresponding OPS.
**The general theory of orthogonal polynomials**

**Existence of OPS**

**Theorem:** Let $\mathcal{L}$ be the functional associated to the sequence of moments $(\mu_n)_n$ and let $(P_n)_n$ a polynomial sequence.

Then $(P_n)_n$ is an OPS w.r.t. $\mathcal{L}$ iff $\Delta_n \neq 0$, $\forall n$. Moreover the leading coefficient $a_n$ in $P_n(x) = a_n x^n + \cdots$ is given by $a_n = \frac{K_n \Delta_{n-1}}{\Delta_n}$.

**Remark:** Not any sequence $(\mu_n)_n$ defines an OPS. An example is the sequences $(\mu_k)_k$ s.t. $\mu_0 = 0$, or $\mu_0 = \mu_1 = \mu_2$ (why?)

If $\mathcal{L}$ has an integral representation $\mathcal{L}[f] = \int_a^b f(x) \rho(x) dx$, $\rho(x) > 0$, then it can be proved that there exists the corresponding OPS.

**Definition:** A moment functional $\mathcal{L}$ is say to be **positive definite** if

for all non-negative polynomials $\pi$, and $\pi \not\equiv 0$ on $\mathbb{R}$ ($\pi(x) \geq 0$) $\mathcal{L}[\pi] > 0$

**Theorem:** Let $\mathcal{L}$ be positive definite moment functional.

Then all the moments $\mu_n$ of $\mathcal{L}$ are real and there exists an OPS of real polynomials associated with $\mathcal{L}$. 
Some consequences of the orthogonality relation

**Theorem:** The orthogonal polynomials \((P_n)_n\) satisfy a three-term recurrence relation of the form:

\[
XP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),
\]

where \(\alpha_n, \beta_n\) and \(\gamma_n\) are the numerical sequences given by

\[
\alpha_n = \frac{\mathcal{L}[xP_nP_{n+1}]}{\mathcal{L}[P_{n+1}^2]}, \quad \beta_n = \frac{\mathcal{L}[xP_nP_n]}{\mathcal{L}[P_n^2]}, \quad \gamma_n = \frac{\mathcal{L}[xP_nP_{n-1}]}{\mathcal{L}[P_{n-1}^2]}.
\]

Usually, \(P_{-1}(x) = 0\) and \(P_0(x) = 1\), thus \((P_n)_n\) is uniquely defined by the numerical sequences \((\alpha_n)_n\), \((\beta_n)_n\) and \((\gamma_n)_n\).
Theorem: The orthogonal polynomials \((P_n)_n\) satisfy a three-term recurrence relation of the form:

\[ xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \]

where \(\alpha_n\), \(\beta_n\) and \(\gamma_n\) are the numerical sequences given by

\[ \alpha_n = \frac{\mathcal{L}[xP_nP_{n+1}]}{\mathcal{L}[P_{n+1}^2]}, \quad \beta_n = \frac{\mathcal{L}[xP_nP_n]}{\mathcal{L}[P_n^2]}, \quad \gamma_n = \frac{\mathcal{L}[xP_nP_{n-1}]}{\mathcal{L}[P_{n-1}^2]}. \]

Usually, \(P_{-1}(x) = 0\) and \(P_0(x) = 1\), thus \((P_n)_n\) is uniquely defined by the numerical sequences \((\alpha_n)_n\), \((\beta_n)_n\) and \((\gamma_n)_n\).

The converse is also true!
Favard’s Theorem:

Let be \((\beta_n)_{n=0}^{\infty}\) and \((\gamma_n)_{n=0}^{\infty}\) two sequences of real numbers and let \((P_n)_n\) be a sequence of monic polynomials satisfying the TRRR

\[ P_n(x) = (x - \beta_{n-1})P_{n-1}(x) - \gamma_{n-1}P_{n-2}(x), \quad n = 1, 2, 3, \ldots \]

If \(\gamma_{n-1} \neq 0\), for all \(n \geq 1\), then there exists a unique moment functional \(\mathcal{L}\) such that

\[ \mathcal{L}[1] = \gamma_0, \quad \mathcal{L}[P_n P_m] = 0 \quad \text{if} \quad n \neq m. \]

Moreover, \(\mathcal{L}\) is positive definite iff \(\gamma_n > 0, \forall n = 0, 1, 2, \ldots\)
Let define $\mathcal{L}$ by induction on $\mathbb{P}_n$ in the following way:

$$\mathcal{L}[1] = \mu_0 = \gamma_0, \quad \mathcal{L}[P_n] = 0, \quad n = 1, 2, 3, \ldots$$

Then, using the TTRR we can calculate all $\mu_n$ as follows: Since $\mathcal{L}[P_n] = 0$,

$$0 = \mathcal{L}[P_1] = \mathcal{L}[x - \beta_0] = \mu_1 - \beta_0 \gamma_0, \quad \text{then} \quad \mu_1 = \beta_0 \gamma_0.$$  

$$0 = \mathcal{L}[P_2] = \mathcal{L}[(x - \beta_1)P_1 - \gamma_1 P_0] = \mu_2 - (\beta_0 + \beta_1)\mu_1 + (\beta_0 \beta_1 - \gamma_1)\gamma_0,$$

from where $\mu_2$ follows, and so on. Next, we use the TTRR and Eq. in blue:

$$x^k P_n(x) = \sum_{i=n-k}^{n+k} d_{n,i} P_i(x).$$

Hence, $\mathcal{L}[x^k P_n] = 0, \forall k = 1, 2, \ldots, n - 1$. Finally,

$$\mathcal{L}[x^n P_n] = \mathcal{L}[x^{n-1}(P_{n+1} + \beta_n P_n + \gamma_n P_{n-1})] = \gamma_n \mathcal{L}[x^{n-1}P_{n-1}] \quad \Rightarrow$$

$$\mathcal{L}[x^n P_n] = \gamma_n \gamma_{n-1} \cdots \gamma_0 \neq 0.$$

Then $\mathcal{L}$ is positive definite (so then the MOPS $(P_n)_n$) iff $\forall n \geq 0, \gamma_n > 0.$
The general theory of orthogonal polynomials

The Christoffel-Darboux formula

**Theorem:** Let \((P_n)_n\) an OPS satisfying the TTRR, then

\[
\text{Ker}_n(x, y) := \sum_{m=0}^{n} \frac{P_m(x)P_m(y)}{d_m^2} = \frac{\alpha_n}{d_n^2} \frac{P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)}{x - y}, \quad n \geq 1.
\]
The general theory of orthogonal polynomials

The Christoffel-Darboux formula

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\]

To prove it we write the TTRR

\[
xP_k(x) = \alpha_k P_{k+1}(x) + \beta_k P_k(x) + \gamma_k P_{k-1}(x),
\]

\[
yP_k(y) = \alpha_k P_{k+1}(y) + \beta_k P_k(y) + \gamma_k P_{k-1}(y).
\]

Then, multiplying the 1st Eq. by \(P_k(y)\), the 2nd one by \(P_k(x)\) and substrating both we find

\[
(x - y) \frac{P_k(x)P_k(y)}{d_k^2} = A_k - A_{k-1},
\]

where

\[
A_k = \frac{\alpha_k}{d_k^2} [P_{k+1}(x)P_k(y) - P_{k+1}(y)P_k(x)].
\]

Hence, summing from \(k = 0\) to \(k = n\) the result follows.
Corollary: The confluent Christoffel-Darboux formula holds:

\[
\text{Ker}_n(x, x) = \sum_{m=0}^{n} \frac{P_m^2(x)}{d_m^2} = \frac{\alpha_n}{d_n^2} \left[ P_{n+1}'(x)P_n(x) - P_{n+1}(x)P_n'(x) \right] \quad n \geq 1.
\]
Corollary: The confluent Christoffel-Darboux formula holds:

$$\text{Ker}_n(x, x) = \sum_{m=0}^{n} \frac{P_m^2(x)}{d_m^2} = \frac{\alpha_n}{d_n^2} [P_{n+1}'(x)P_n(x) - P_{n+1}(x)P_n'(x)] \quad n \geq 1.$$ 

In the following we assume that $\mathcal{L}[f] = \int_a^b f(x) d\mu(x)$, $\mu$ a positive Borel measure in $(a, b)$.

Theorem: Let $(P_n)_n$ an OPS satisfying w.r.t. a positive definite $\mathcal{L}$

1. Two consecutive polynomials $P_n$ and $P_{n+1}$ can not have common zeros.
2. All the zeros of $P_n$ are real, simple and are located inside the interval $(a, b)$.
3. Denote by $x_{n,j}$, $j = 1, 2, \ldots, n$ the zeros of $P_n$, and assume that they are ordered in an increasing way $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$. Then, the zeros of $P_n$ and $P_{n+1}$ mutually separate each other, i.e.,

$$x_{n+1,j} < x_{n,j} < x_{n+1,j+1} < x_{n,j+1}.$$
The TTRR can be written in matrix form

\[ xP_{n-1} = J_n P_{n-1} + \alpha_n P_n(x) e_n \]

where

\[
P_{n-1} = \begin{bmatrix}
P_0(x) \\
P_1(x) \\
P_2(x) \\
\vdots \\
P_{n-2}(x) \\
P_{n-1}(x)
\end{bmatrix}, \quad J_n = \begin{bmatrix}
\beta_1 & \alpha_1 & 0 & \ldots & 0 & 0 \\
\gamma_2 & \beta_2 & \alpha_2 & \ldots & 0 & 0 \\
0 & \gamma_3 & \beta_3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \beta_{n-1} & \alpha_{n-1} \\
0 & 0 & 0 & \ldots & \gamma_n & \beta_n
\end{bmatrix}, \quad e_n = \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}.
\]

Denoting by \( \{x_{n,j}\}_{1 \leq j \leq n} \) the zeros of the polynomial \( P_n \), we see from the TTRR in matrix form that each \( x_{n,j} \) is an eigenvalue of the corresponding tridiagonal matrix of order \( n \) and \[ P_0(x_{n,j}), \ldots, P_{n-1}(x_{n,j}) \]^T is the associated eigenvector.
In the special case of orthonormal polynomials $J_n$ is the Jacobi matrix

$$J_n = \begin{bmatrix}
\beta_1 & \alpha_1 & 0 & \ldots & 0 & 0 \\
\alpha_2 & \beta_2 & \alpha_2 & \ldots & 0 & 0 \\
0 & \alpha_3 & \beta_3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \beta_{n-1} & \alpha_{n-1} \\
0 & 0 & 0 & \ldots & \alpha_n & \beta_n
\end{bmatrix}$$

**Proposition:** Let $J_n$ a Jacobi matrix. Then:

1. The eigenvalues of $J_n$ are real and simple.
2. The eigenvalues of $J_n$ and $J_{n+1}$ interlace.
The general theory of orthogonal polynomials

Properties of zeros of OPs

In the special case of orthonormal polynomials $J_n$ is the Jacobi matrix

$$J_n = \begin{bmatrix}
\beta_1 & \alpha_1 & 0 & \ldots & 0 & 0 \\
\alpha_2 & \beta_2 & \alpha_2 & \ldots & 0 & 0 \\
0 & \alpha_3 & \beta_3 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \beta_{n-1} & \alpha_{n-1} \\
0 & 0 & 0 & \ldots & \alpha_n & \beta_n
\end{bmatrix}$$

**Proposition:** Let $J_n$ a Jacobi matrix. Then:

1. The eigenvalues of $J_n$ are real and simple.
2. The eigenvalues of $J_n$ and $J_{n+1}$ interlace.

**Theorem:** Let $P_n$ and $P_{n-1}$ be two monic pol. of deg. $n$ & $n-1$, resp.

Let $a < x_1 < x_2 < \cdots < x_n < b$ and $y_1 < y_2 < \cdots < y_{n-1}$ the real zeros of $P_n$ and $P_{n-1}$, resp., and suppose that they interlace, i.e., $x_i < y_i < x_{i+1}$, $i = 1, 2, 3, \ldots, n-1$. Then $\exists (P_k)_{k=0}^n$ orthogonal on $(a, b)$ s.t. the above polynomials $P_n$ and $P_{n-1}$ belong to it.
The general theory of orthogonal polynomials

The quadrature formula

Let be the positive measure \( d\mu(x) \) supported on \((a, b)\) and let \((P_n)_n\) the corresponding OPS. Given and integrable function we want to calculate the integral of \( f \) using the following quadrature formula (QF)

\[
\int_a^b f(x) d\mu(x) = \sum_{k=1}^{n} \lambda_k f(x_k) + R_n(f), \quad \lambda_k \in \mathbb{R}, \quad k = 1, 2, \ldots, n,
\]

s.t. \( R_n(\pi) = 0 \), for all \( \pi \in \mathbb{P}_N \). \( N \) is called de degree of exactness of the QF and numbers \( x_1 < x_2 < \cdots < x_n \) are the nodes of the QF.

The Gauss QF is a QF such that \( R_n(\pi) = 0 \) for all \( \pi \in \mathbb{P}_{2n-1} \) and the nodes are the zeros of \( P_n \) which belongs to the MOPS w.r.t. \( \mathcal{L}[f] = \int_a^b f(x) d\mu(x) \).

Idea: Use \( \pi(x) = \frac{P_n(x)P_{n-1}(x)}{(x - x_j)} \) in GQF \( \Rightarrow \)

\[
\lambda_j = \frac{1}{P_n'(x_j)P_{n-1}(x_j)} \int_a^b \frac{P_n(x)}{(x - x_j)} P_{n-1}(x) d\mu(x)
\]
The general theory of orthogonal polynomials

The quadrature formula

Let be the positive measure $d\mu(x)$ supported on $(a, b)$ and let $(P_n)_n$ the corresponding OPS. Given and integrable function we want to calculate the integral of $f$ using the following quadrature formula (QF)

$$\int_a^b f(x) d\mu(x) = \sum_{k=1}^n \lambda_k f(x_k) + R_n(f), \quad \lambda_k \in \mathbb{R}, \quad k = 1, 2, \ldots, n,$$

s.t. $R_n(\pi) = 0$, for all $\pi \in \mathbb{P}_N$. $N$ is called de degree of exactness of the QF and numbers $x_1 < x_2 < \cdots < x_n$ are the nodes of the QF.

The Gauss QF is a QF such that $R_n(\pi) = 0$ for all $\pi \in \mathbb{P}_{2n-1}$ and the nodes are the zeros of $P_n$ which belongs to the MOPS w.r.t. $\mathcal{L}[f] = \int_a^b f(x) d\mu(x)$.

Idea: Use $\pi(x) = \frac{P_n(x)P_{n-1}(x)}{(x - x_j)}$ in GQF $\Rightarrow \frac{P_n(x)}{(x - x_j)} = P_{n-1}(x) + q_{n-2}(x)$

$$\lambda_j = \frac{1}{P'_n(x_j)P_{n-1}(x_j)} \int_a^b \frac{P_n(x)}{(x - x_j)} P_{n-1}(x) d\mu(x) = \frac{1}{\text{Ker}_{n-1}(x_k, x_k)} > 0.$$
Constructive approximation and special functions: theory and applications

Renato Álvarez-Nodarse

Sevilla, March 2010
The solution of the hypergeometric equation

The hypergeometric property

**Definition:** We call the equation

\[ \sigma(z)y'' + \tau(z)y' + \lambda y = 0, \]

the equation of hypergeometric type, and its solutions the functions of hypergeometric type.
The hypergeometric property

**Definition:** We call the equation

\[ \sigma(z)y'' + \tau(z)y' + \lambda y = 0, \]

the equation of hypergeometric type, and its solutions the functions of hypergeometric type.

**Theorem:** All the derivatives \( y^{(m)} \equiv y_m \) of the functions of hypergeometric type \( y \) are also functions of hypergeometric type, i.e., they satisfy (HED)

\[ \sigma(x)y_m'' + \tau_m(x)y_m' + \mu_m y_m = 0, \]

\[ \tau_m(x) = \tau(x) + m\sigma'(x), \quad \mu_m = \lambda + m\tau'(x) + \frac{m(m-1)}{2}\sigma''(x). \]

The proof is by induction.
Proof of the *hypergeometric* property

Taking the 1st derivative in \( \sigma(z)y'' + \tau(z)y' + \lambda y = 0 \)

\[
\sigma y_1'' + [\tau(x) + \sigma'(x)]y_1' + (\lambda + \tau')y_1 = 0.
\]
Proof of the \textit{hypergeometric} property

Taking the 1st derivative in \( \sigma(z)y'' + \tau(z)y' + \lambda y = 0 \)

\[ \sigma y''_1 + [\tau(x) + \sigma'(x)]y'_1 + (\lambda + \tau')y_1 = 0. \]

Let \( \tau_1(x) = \tau(x) + \sigma'(x) \) & \( \mu_1 = \lambda + \tau' \), \( \Rightarrow \)

\[ \sigma y''_1 + \tau_1(x)y'_1 + \mu_1 y_1 = 0, \quad \deg \sigma \leq 2, \quad \deg \tau_1 \leq 1. \]
Proof of the hypergeometric property

Taking the 1st derivative in \( \sigma(z)y'' + \tau(z)y' + \lambda y = 0 \)

\[ \sigma y_1'' + [\tau(x) + \sigma'(x)]y_1' + (\lambda + \tau')y_1 = 0. \]

Let \( \tau_1(x) = \tau(x) + \sigma'(x) \) & \( \mu_1 = \lambda + \tau' \), \( \Rightarrow \)

\[ \sigma y_1'' + \tau_1(x)y_1' + \mu_1y_1 = 0, \quad \text{deg } \sigma \leq 2, \quad \text{deg } \tau_1 \leq 1. \]

Suppose that \( y_{m-1} \) satisfies \( \sigma(x)y_{m-1}'' + \tau_{m-1}(x)y_{m-1}' + \mu_{m-1}y_{m-1} = 0 \).

Then by taking derivatives

\[ \sigma(x)y_m'' + \tau_m(x)y_m' + \mu_my_m = 0, \]

\[ \tau_m(x) = \tau_{m-1}(x) + \sigma'(x), \quad \mu_m = \mu_{m-1} + \tau'_{m-1}. \]
Proof of the hypergeometric property

Taking the 1st derivative in $\sigma(z)y'' + \tau(z)y' + \lambda y = 0$

$\sigma y'' + [\tau(x) + \sigma'(x)]y' + (\lambda + \tau')y_1 = 0.$

Let $\tau_1(x) = \tau(x) + \sigma'(x)$ & $\mu_1 = \lambda + \tau'$, \implies

$\sigma y'' + \tau_1(x)y' + \mu_1 y_1 = 0, \quad \text{deg } \sigma \leq 2, \quad \text{deg } \tau_1 \leq 1.$

Suppose that $y_{m-1}$ satisfies $\sigma(x)y''_{m-1} + \tau_{m-1}(x)y'_{m-1} + \mu_{m-1}y_{m-1} = 0.$

Then by taking derivatives

$\sigma(x)y'' + \tau_m(x)y' + \mu_my_m = 0,$

$\tau_m(x) = \tau_{m-1}(x) + \sigma'(x), \quad \mu_m = \mu_{m-1} + \tau'_{m-1}.$

$\tau_m(x) = \tau_{m-1}(x) + \sigma'(x) = \tau_{m-2}(x) + 2\sigma'(x) = \cdots = \tau_0(x) + m\sigma'(x), \quad \tau_0(x) = \tau(x).$

$\mu_m - \mu_{m-1} = \tau'_{m-1}, \quad \implies \sum_{k=1}^{m}(\mu_k - \mu_{k-1}) = \sum_{k=1}^{m} \tau'_{k-1} \implies$
Proof of the \textit{hypergeometric} property

Taking the 1st derivative in $\sigma(z)y'' + \tau(z)y' + \lambda y = 0$

$$\sigma y''_1 + [\tau(x) + \sigma'(x)]y'_1 + (\lambda + \tau')y_1 = 0.$$  

Let $\tau_1(x) = \tau(x) + \sigma'(x)$ & $\mu_1 = \lambda + \tau'$, $\Rightarrow$

$$\sigma y''_1 + \tau_1(x)y'_1 + \mu_1 y_1 = 0, \quad \deg\sigma \leq 2, \quad \deg\tau_1 \leq 1.$$  

Suppose that $y_{m-1}$ satisfies $\sigma(x)y''_{m-1} + \tau_{m-1}(x)y'_{m-1} + \mu_{m-1}y_{m-1} = 0$. Then by taking derivatives

$$\sigma(x)y''_m + \tau_m(x)y'_m + \mu_my_m = 0,$$

$$\tau_m(x) = \tau_{m-1}(x) + \sigma'(x), \quad \mu_m = \mu_{m-1} + \tau'_{m-1}.$$  

$$\tau_m(x) = \tau_{m-1}(x) + \sigma'(x) = \tau_{m-2}(x) + 2\sigma'(x) = \cdots = \tau_0(x) + m\sigma'(x), \quad \tau_0(x) = \tau(x).$$  

$$\mu_m - \mu_{m-1} = \tau'_{m-1}, \quad \Rightarrow \quad \sum_{k=1}^{m} (\mu_k - \mu_{k-1}) = \sum_{k=1}^{m} \tau'_{k-1} \quad \Rightarrow$$

$$\mu_m - \mu_0 = \sum_{k=1}^{m} \tau'_{k-1}, \quad \mu_0 = \lambda, \quad \tau'_{k-1} = \tau' + (k - 1)\sigma''$$

The inverse is also true. We leaves this as an exercise.
The solution of the hypergeometric equation

The classical orthogonal polynomials

The Rodrigues formula

The (HE) and (HED) can be written in their self-adjoint form

\[ [\sigma(x)\rho(x)y']' + \lambda \rho(x)y = 0, \quad [\sigma(x)\rho_m(x)y_m']' + \mu_m \rho_m(x)y_m = 0, \]

where \( \rho \) and \( \rho_m \) are the the solutions of the son Pearson type differential equations

\[ [\sigma(x)\rho(x)]' = \tau(x)\rho(x), \quad [\sigma(x)\rho_m(x)]' = \tau_m(x)\rho_m(x). \]

Notice that from \( \tau_m(x) = \tau(x) + m\sigma'(x) \) it follows that \( \rho_m(x) = \sigma^m(x)\rho(x) \).

**Theorem:** The pol. solutions of (HED) are given by the Rodrigues formula:

\[ P_n^{(m)}(x) = \frac{A_{nm}B_n}{\rho_m(x)} \frac{d^{n-m}}{dx^{n-m}}[\rho_n(x)], \quad B_n = P_n^{(n)}/A_{nn} \]

where \( A_{nm} = A_m(\lambda) \bigg|_{\lambda=\lambda_n} = \frac{n!}{(n-m)!} \prod_{k=0}^{m-1} \left[ \tau' + \frac{1}{2}(n + k - 1)\sigma'' \right] \) and

\[ \mu_m = \mu_m(\lambda_n) = -(n - m)[\tau' + \frac{1}{2}(n + m - 1)\sigma'']. \]
The solution of the hypergeometric equation
The classical orthogonal polynomials

The Rodrigues formula

As a Corollary we have

**Theorem:** The pol. solutions of (HE) are given by the Rodrigues formula:

\[ P_n(x) = \frac{B_n}{\rho(x)} \frac{d^n}{dx^n} [\rho_n(x)], \quad B_n = \frac{P_n^{(n)}}{A_{nn}} \]

where

\[ \lambda := \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma''. \]

If \( n = 1 \) we obtain

\[ P_1(x) = \frac{B_1}{\rho(x)} [\sigma(x)\rho(x)]' = B_1\tau(x), \quad \implies \quad \text{deg} \ \tau = 1. \]

Notice that \( P_1(s) \) and \( \tau(x) \) have the same root.
The solution of the hypergeometric equation

The classical orthogonal polynomials

The orthogonality relation

**Theorem:**

If \( x^k \sigma(x) \rho(x) \bigg|_a^b = 0 \), for all \( k \geq 0 \),

Then, the pol. solutions of (HE) \( \sigma(z)y''' + \tau(z)y' + \lambda y = 0 \) are orthogonal w.r.t. the weight function \( \rho \) which is the solution of the Pearson Eq.

\[
\sigma(x) \rho(x) \big[ \sigma(x) \rho(x) \big]' = \tau(x) \rho(x).
\]

I.e., the following holds

\[
\int_a^b P_n(x) P_m(x) \rho(x) dx = \delta_{nm} d_n^2,
\]

where \( \delta_{nm} \) is the Kronecker symbol and \( d_n \) denotes the norm of \( P_n \).

In a similar way buy using (EHD) and the Pearson equation for \( \rho_k \) obtain

\[
\int_a^b P_n^{(k)}(x) P_m^{(k)}(x) \rho_k(x) dx = \delta_{nm} d_{kn}^2.
\]
The solution of the hypergeometric equation

Some consequences of the orthogonality relation

**Theorem:** The orthogonal polynomials \((P_n)_n\) satisfy a three-term recurrence relation of the form:

\[ xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \]

where \(\alpha_n\), \(\beta_n\) and \(\gamma_n\) are some numerical sequences. In fact, if \(a_n, b_n,\) and \(c_n\) are the coefficients in the expansion \(P_n(x) = a_n x^n + b_n x^{n-1} + c_n x^{n-2} + \cdots\), and \(d_n\) the norms of \(P_n\), then

\[ \alpha_n = \frac{a_n}{a_{n+1}}, \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \quad \gamma_n = \frac{c_n - \alpha_n c_{n+1}}{a_{n-1}} - \frac{b_n}{a_{n-1}} \beta_n = \frac{a_{n-1}}{a_n} \frac{d_{n}^2}{d_{n-1}^2}. \]

Usually, \(P_{-1}(x) = 0\) and \(P_0(x) = 1\), thus \((P_n)_n\) is uniquely defined by the numerical sequences \((\alpha_n)_n\), \((\beta_n)_n\) and \((\gamma_n)_n\).
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where \(\alpha_n\), \(\beta_n\) and \(\gamma_n\) are some numerical sequences. In fact, if \(a_n\), \(b_n\), and \(c_n\) are the coefficients in the expansion \(P_n(x) = a_n x^n + b_n x^{n-1} + c_n x^{n-2} + \cdots\), and \(d_n\) the norms of \(P_n\), then

\[
\alpha_n = \frac{a_n}{a_{n+1}}, \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \quad \gamma_n = \frac{c_n - \alpha_n c_{n+1}}{a_{n-1}} - \frac{b_n}{a_{n-1}} \beta_n = \frac{a_{n-1}}{a_n} \frac{d_n^2}{d_{n-1}^2}.
\]

Usually, \(P_{-1}(x) = 0\) and \(P_0(x) = 1\), thus \((P_n)\) is uniquely defined by the numerical sequences \((\alpha_n)_n\), \((\beta_n)_n\) and \((\gamma_n)_n\).

Remark: The inverse is also true:

Given a sequence \((P_n)\) satisfying the TRRR such that \(\alpha_n \gamma_n > 0\) and \(\beta_n \in \mathbb{R}\), then there exists a measure such that the \((P_n)\) is and OPS. This result is usually called Favard Theorem.
Some consequences of the orthogonality relation

Substituting the Rodrigues formula in the orthogonality relation and integrating by parts we obtain

\[ d_n^2 = B_n (-1)^n n! a_n \int_a^b \sigma^n(x) \rho(x) dx. \]
Substituting the Rodrigues formula in the orthogonality relation and integrating by parts we obtain

\[ d_n^2 = B_n(-1)^n n! a_n \int_a^b \sigma^n(x) \rho(x) \, dx. \]

**Remark:** Suppose that \( \sigma(x) > 0 \) in the orthogonality interval \((a, b)\), and we want to assure the weight function to be positive and integrable in \((a, b)\), then \( \tau \) must satisfy the following two conditions:

1. \( \tau' < 0 \). This quite important for the case when \( \sigma = 1 \) and \( \sigma = x \), since in these two cases the orthogonality interval are not bounded.
2. \( \tau \) should has its zero inside \((a, b)\). It follows from the fact that \( P_1(x) = B_1 \tau(x) \).
The solution of the hypergeometric equation

Some consequences of the Rodrigues formula

Choosing \( m = 1 \) in \( P_n^{(m)}(x) = \frac{A_{nm} B_n}{\rho_m(x)} \frac{d^{n-m}}{dx^{n-m}}[\rho_n(x)] \) and doing some straightforward calculations we have

\[
P'_n(x) = \frac{A_{n1} B_n}{\rho_1(x)} \frac{d^{n-1}}{dx^{n-1}}[\rho_n(x)] = -\frac{\lambda_n B_n}{\rho_1(x)} \frac{d^{n-1}}{dx^{n-1}}[\rho_{1_{n-1}}(x)] \quad \Rightarrow
\]

\[
P'_n(x) = \frac{-\lambda_n B_n}{B_{n-1}} \bar{P}_{n-1}(x),
\]

where \( \bar{P}_{n-1} \) is the polynomial orthogonal w.r.t. \( \rho_1(x) = \sigma(x) \rho(x) \).

**Theorem:**

The OP solutions of (HE) \((P_n)_n\) are such that their derivatives \((P_n^{(m)})_n\) are also orthogonal.

The converse is also true. This leads to the

**Hahn’s Theorem:**

A given sequence of orthogonal polynomials \((P_n)_n\) satisfy the (HE) iff the sequence of its derivatives \((P'_n)_n\) is an orthogonal polynomial sequence.
Some consequences of the Rodrigues formula

Writting \( P_{n+1}(x) = \frac{B_{n+1}}{\rho(x)} \frac{d^{n+1}}{dx^{n+1}}[\sigma^{n+1}(x)\rho(x)] = \frac{B_{n+1}}{\rho(x)} \frac{d^n}{dx^n}[\tau_n(x)\rho_n(x)] = \)

\[ \frac{B_{n+1}}{\rho(x)} \left[ \tau_n(x) \frac{d^n \rho_n(x)}{dx^n} + n\tau'_n \frac{d^{n-1} \rho_n(x)}{dx^{n-1}} \right]. \]

But \( P'_n(x) = \frac{-\lambda_n B_n}{\sigma(x)\rho(x)} \frac{d^{n-1} \rho_n(x)}{dx^{n-1}} \), thus

\[ \sigma(x)P'_n(x) = \frac{\lambda_n}{n\tau'_n} \left[ \tau_n(x)P_n(x) - \frac{B_n}{B_{n+1}} P_{n+1}(x) \right]. \]

The last formula is the differentiation formula for the classical OP. Using the TTRR we obtain the structure formula of Al-Salam and Chihara

\[ \sigma(x)P'_n(x) = \tilde{\alpha}_n P_{n+1}(x) + \tilde{\beta}_n P_n(x) + \tilde{\gamma}_n P_{n-1}(x), \quad n \geq 0. \]
Suppose that $\rho_n(z) = \rho(z)\sigma^n(z)$ is an analytic function on a domain $\Omega$ and let $C \subset \Omega$ be a closed smooth curve surrounding the point $z = x$. Then, by the integral Cauchy formula

$$\rho_n(x) = \frac{1}{2\pi i} \int_C \frac{\rho_n(z)}{z-x} \, dz \quad \Rightarrow$$

$$P_n(x) = \frac{B_n}{2\pi i \rho(x)} \frac{d^n}{dz^n} \int_C \frac{\rho_n(z)}{z-x} \, dz = \frac{n! B_n}{2\pi i \rho(x)} \int_C \frac{\rho_n(z)}{(z-x)^{n+1}} \, dz.$$  

This is extremely important formula for applications. In particular from it follows the important

**Theorem:** Any three classical polynomials $P_n^{(k_i)}$ are connected by the linear relation

$$A_1(x)P_{n_1}^{(k_1)}(x) + A_2(x)P_{n_2}^{(k_2)}(x) + A_3(x)P_{n_3}^{(k_3)}(x) = 0,$$

where $A_i$ are polynomials on $x$ of degree independent of $n_1, n_2, n_3$.

**This is leaved as an exercise.**
The solution of the hypergeometric equation

The classical orthogonal polynomials

**The Hermite, Laguerre, and Jacobi polynomials**

<table>
<thead>
<tr>
<th>( P_n(x) )</th>
<th>( H_n(x) )</th>
<th>( L_n^\alpha(x) )</th>
<th>( P_n^\alpha,\beta(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma(x) )</td>
<td>1</td>
<td>( x )</td>
<td>( 1 - x^2 )</td>
</tr>
<tr>
<td>( \tau(x) )</td>
<td>(-2x)</td>
<td>(-x + \alpha + 1)</td>
<td>(-(\alpha + \beta + 2)x + \beta - \alpha)</td>
</tr>
<tr>
<td>( \lambda_n )</td>
<td>( 2n )</td>
<td>( n )</td>
<td>( n(n + \alpha + \beta + 1) )</td>
</tr>
<tr>
<td>( \rho(x) )</td>
<td>( e^{-x^2} )</td>
<td>( x^\alpha e^{-x} )</td>
<td>((1 - x)^\alpha (1 + x)^\beta)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \alpha &gt; -1 )</td>
<td>( \alpha, \beta &gt; -1 )</td>
</tr>
<tr>
<td>( B_n )</td>
<td>( \frac{(-1)^n}{2^n} )</td>
<td>( (-1)^n )</td>
<td>( \frac{(-1)^n}{(n + \alpha + \beta + 1)_n} )</td>
</tr>
</tbody>
</table>
The solution of the hypergeometric equation

The classical orthogonal polynomials

The Hermite, Laguerre, and Jacobi polynomials

\[ H_n(x) = \begin{cases} 
(-1)^m \left( \frac{1}{2} \right)_m {}_1F_1 \left( -\frac{m}{2}, x^2 \right), & n = 2m \\
(-1)^m \left( \frac{3}{2} \right)_m x {}_1F_1 \left( -\frac{m}{3}, x^2 \right), & n = 2m + 1 
\end{cases} \]

\[ L_n^\alpha(x) = \frac{(-1)^n \Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)} {}_1F_1 \left( -n, \frac{\alpha + 1}{x} \right), \]

\[ P_n^{\alpha,\beta}(x) = \frac{2^n(n + 1)_n}{(n + \alpha + \beta + 1)_n} {}_2F_1 \left( -n, n + \alpha + \beta + 1, \frac{1 - x}{2} \right). \]
The solution of the hypergeometric equation

The classical orthogonal polynomials

The Hermite, Laguerre, and Jacobi polynomials

$$H_n(x) = \begin{cases} (-1)^m \left( \frac{1}{2} \right)_m {}_1F_1 \left( \frac{-m}{2} \mid x^2 \right), & n = 2m \\ (-1)^m \left( \frac{3}{2} \right)_m x {}_1F_1 \left( \frac{-m}{3} \mid x^2 \right), & n = 2m + 1 \end{cases}$$

$$L^\alpha_n(x) = \frac{(-1)^n \Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)} {}_1F_1 \left( \frac{-n}{\alpha + 1} \mid x \right),$$

$$P^\alpha,\beta_n(x) = \frac{2^n(\alpha + 1)_n}{(n + \alpha + \beta + 1)_n} {}_2F_1 \left( \frac{-n, n + \alpha + \beta + 1}{\alpha + 1} \mid \frac{1 - x}{2} \right).$$

$$\sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \cdots (a_p)_k x^k}{(b_1)_k(b_2)_k \cdots (b_q)_k k!}.$$
The Hermite, Laguerre, and Jacobi—as well as the Bessel, $\sigma(x) = x^2$—polynomials are usually called the *classical orthogonal polynomials*.

**Definition:** We say that the orthogonal polynomial sequence (OPS) $(P_n)_n$ is a classical OPS with respect to the weight function $\rho$ if

$$
\int_a^b P_n(x)P_m(x)\rho(x)dx = \delta_{mn}d_n^2,
$$

where $\delta_{mn}$ is the Kronecker symbol $\delta_{mn} = 1$ if $n = m$ and 0 otherwise, $d_n$ is the norm of the polynomial $P_n$, $\rho$ is the solution of the Pearson equation

$$
[\sigma(x)\rho(x)]' = \tau(x)\rho(x),
$$

where $\sigma$ and $\tau$ are fixed polynomials of degrees at most 2 and exactly 1, respectively, such that the following boundary conditions hold

$$
\sigma(a)\rho(a) = \sigma(b)\rho(b) = 0.
$$
The Characterization Theorem: The following are equivalent:

1. \((P_n)_n\) is a clas. orth. pol. seq. (COPS) (Hildebrandt 1931).
2. The sequence of its derivatives \((P'_n)_{n \geq 1}\) is an OPS w.r.t. the weight function \(\rho_1(x) = \sigma(x)\rho(x)\), where \(\rho\) satisfies \((\sigma\rho)' = \tau\rho\).
3. \((P_n)_n\) satisfies the SODE with polynomial coefficients (Bochner 1929)
   \[
   \sigma(x)P''_n(x) + \tau(x)P'_n(x) + \lambda_n P_n(x) = 0,
   \]
   where \(\deg(\sigma) \leq 2, \deg(\tau) = 1, \lambda_n\) is a constant.
4. \((P_n)_n\) can be expressed by the Rodrigues formula (Tricomi 1955)
   \[
   P_n(x) = \frac{B_n}{\rho(x)} \frac{d^n}{dx^n}[\sigma^n(x)\rho(x)].
   \]
5. There exist three sequences of complex numbers \((a_n)_n, (b_n)_n, (c_n)_n\), and a polynomial \(\sigma\), \(\deg(\sigma) \leq 2\), such that (Al-Salam & Chihara 1972)
   \[
   \sigma(x)P'_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \quad n \geq 1.
   \]
6. There exist two sequences of complex numbers \((f_n)_n\) and \((g_n)_n\) such that the following relation for the monic polynomials holds (Marcellán et al. 1994)
   \[
   P_n(x) = \frac{P'_{n+1}(x)}{n+1} + f_n P'_n(x) + g_n P'_{n-1}(x), \quad g_n \neq \gamma_n, \quad n \geq 1,
   \]