A spectral realization of the Riemann zeros

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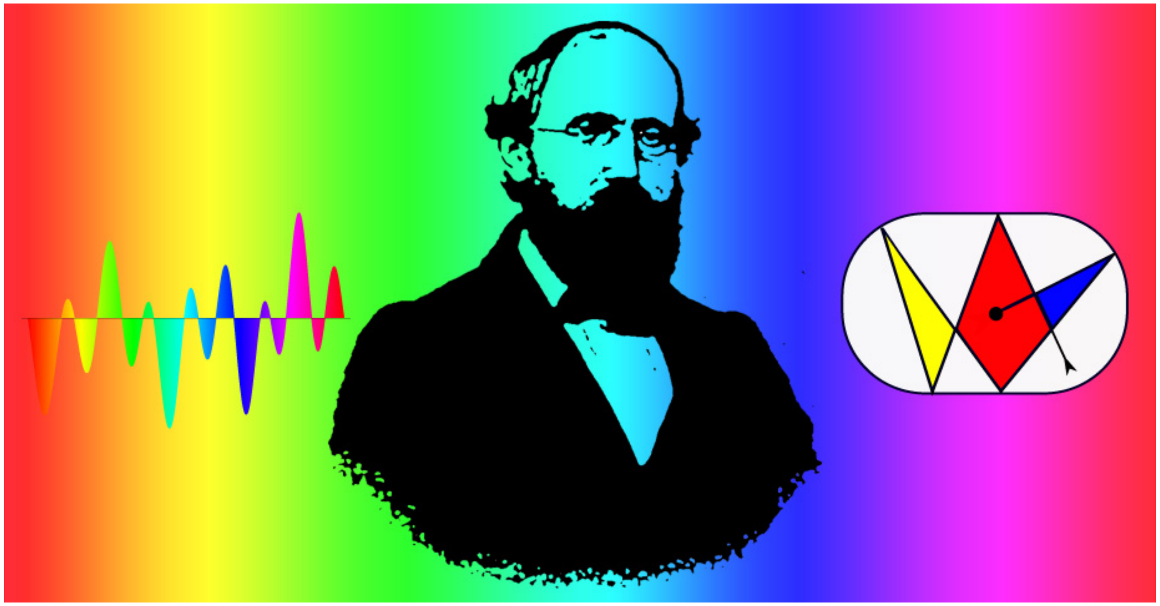
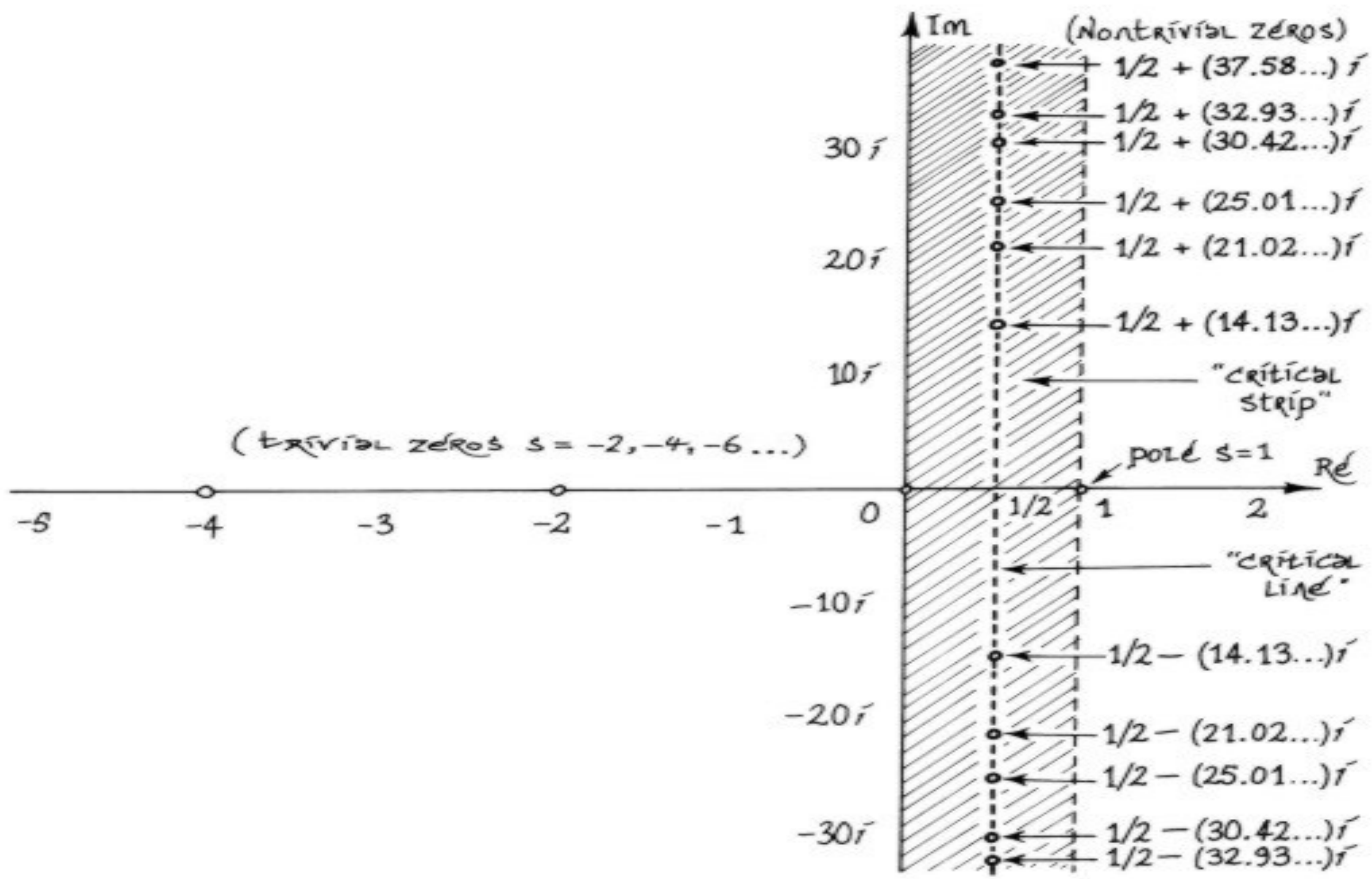


The Riemann Hypothesis (1859)

The complex zeros of the zeta function $\varsigma(s)$ have real part equal to 1/2



Polya and Hilbert Conjecture (1910)

$$s_n = \frac{1}{2} + i\,E_n, \quad E_n = energies$$

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E_n : eigenvalues of a Hamiltonian H $\longrightarrow$ RH true

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$$s_n = \frac{1}{2} + i\,E_n, \quad E_n = energies$$

E_n : eigenvalues of a Hamiltonian H $\longrightarrow$ RH true

$$\det(E - H) \propto \varsigma(1/2 + i\,E)$$

This conjecture inspired the spectral approach to the RH

The problem is to find H: the Riemann operator

Outline

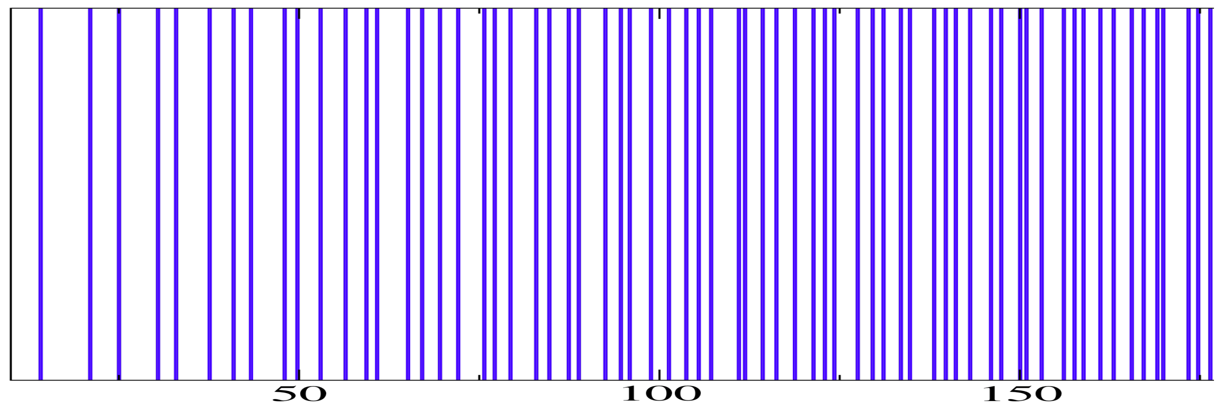
- Hints supporting the Polya-Hilbert conjecture.
- H = xp model of Berry, Keating and Connes
- A modified model H = x(p+ 1/p)
- xp and the Dirac equation
- An ideal optical model of the prime numbers
- Spectral realization of the Riemann zeros

Based on:

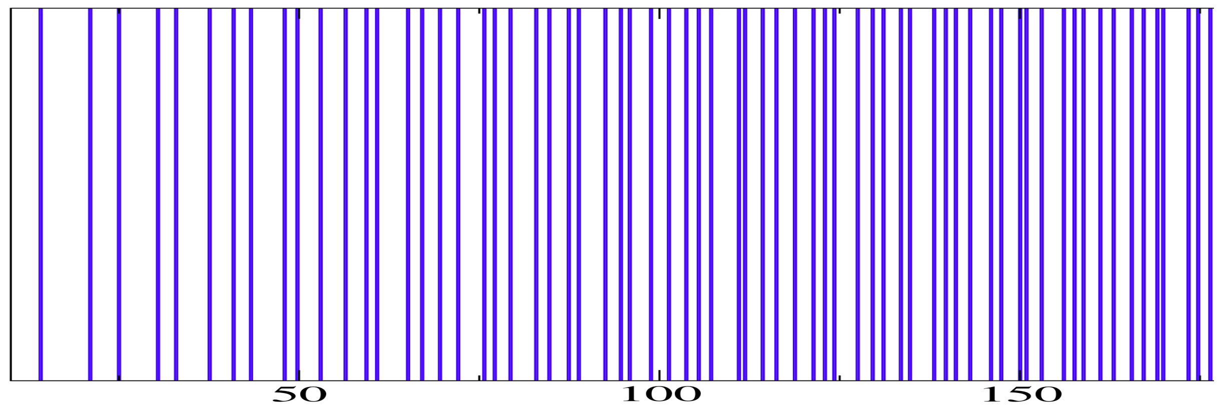
"The H=xp model revisited and the Riemann zeros", (with J. Rodriguez-Laguna) Phys. Rev. Lett. 2011

"The Riemann zeros as energy levels of a Dirac fermion in a potential built from the prime numbers in Rindler spacetime"
J. of Phys. A 2014; arxiv: 1404.4252

The Riemann zeros as spectrum

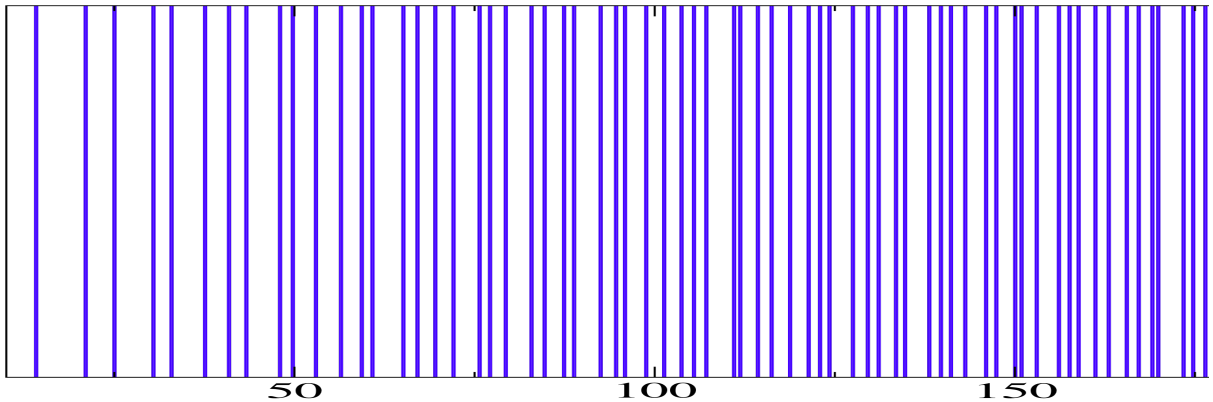


The Riemann zeros as spectrum



$$N_R(E) = \langle N(E) \rangle + N_{fl}(E)$$

The Riemann zeros as spectrum



$$N_R(E) = \langle N(E) \rangle + N_{fl}(E)$$

Average $$\langle N(E) \rangle \approx \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right) + \frac{7}{8} + O(E^{-1})$$

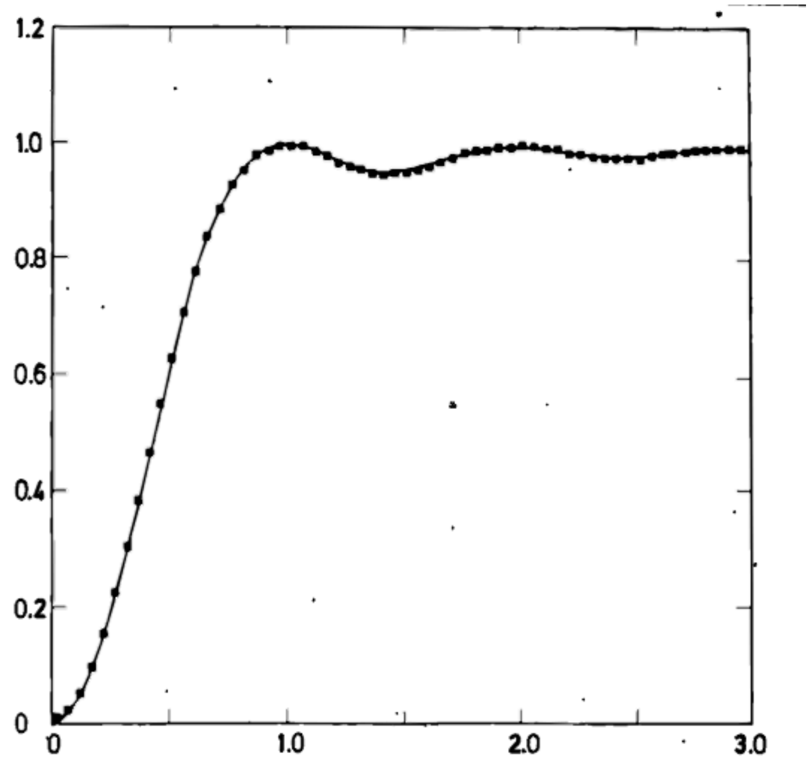
Fluctuation $$N_{fl}(E) = \frac{1}{\pi} Arg\, \varsigma(\frac{1}{2} + i\,E) = O(\log E)$$

Montgomery-Odlyzko law (70-80's)

The fluctuations of the zeros satisfy the Gaussian Unitary Ensemble

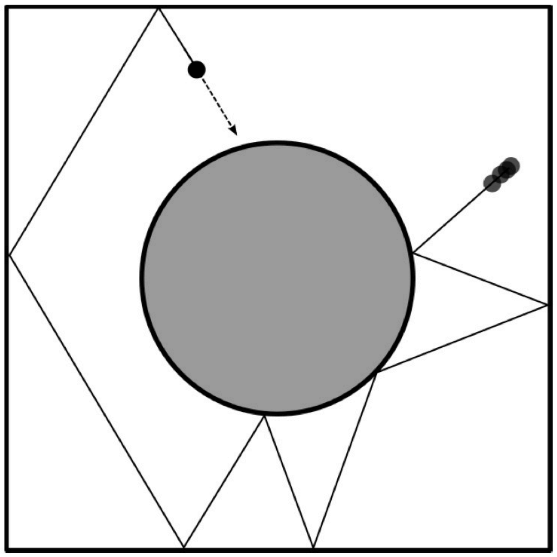
GUE pair correlation function

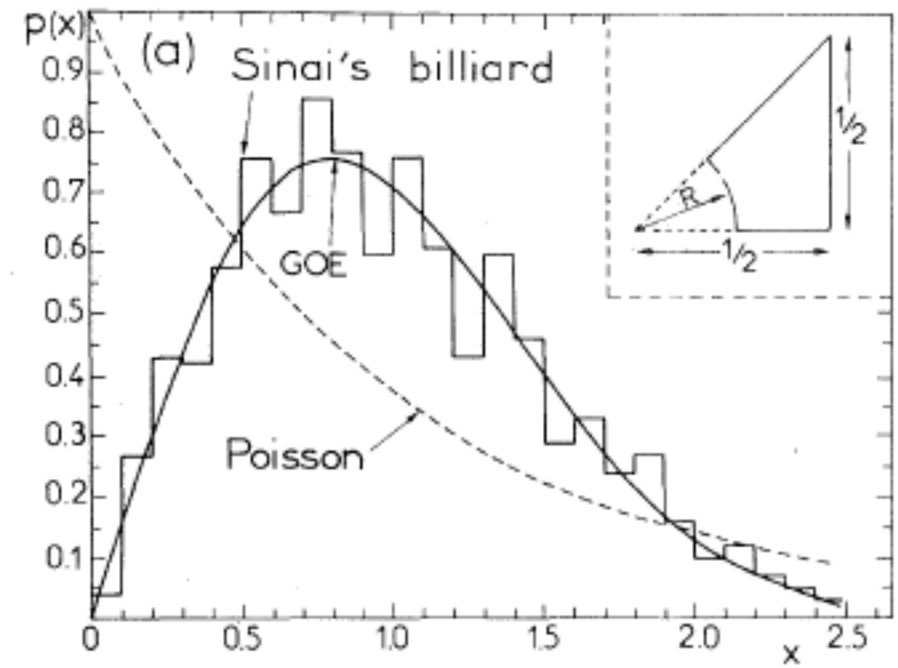
$$R_2(x) = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$$



The zeros behave as the eigenvalues of an random hermitean matrix.
Hence time reversal invariance is broken.

Quantum chaos: Sinai billiard





Bohigas, Giannoni, Schmit (84)

Would the Riemann zeros be the spectrum of a quantum billiard?

Berry´s quantum chaos proposal (80´s):

- H classical chaotic

- H breaks time reversal (GUE – statistics)

- H is quasi-one dimensional

$$\boxed{\begin{array}{l} Periodic\ orbits \leftrightarrow prime\ number\ (p) \\ \qquad Period \quad \leftrightarrow \log p \end{array}}$$

Promise: The quantization of H will give the Riemann zeros

Based on an analogy between the number theory formula

$$N_{fl}(E) = -\frac{1}{\pi}\sum_{p}\sum_{m=1}^{\infty}\frac{1}{m\,p^{m/2}}\sin\left(m\,E\log p\right)$$

and Gutzwiller semiclassical formula

$$N_{fl}(E) = \frac{1}{\pi}\sum_{\alpha}\sum_{m=1}^{\infty}\frac{1}{m\,2\sinh(m\lambda_{\alpha}/2)}\sin\left(m\,E\,T_{\alpha}\right)$$

Log of primes numbers are "periods"

Riemann zeros are "energies"

The Berry-Keating/Connes model (1999):

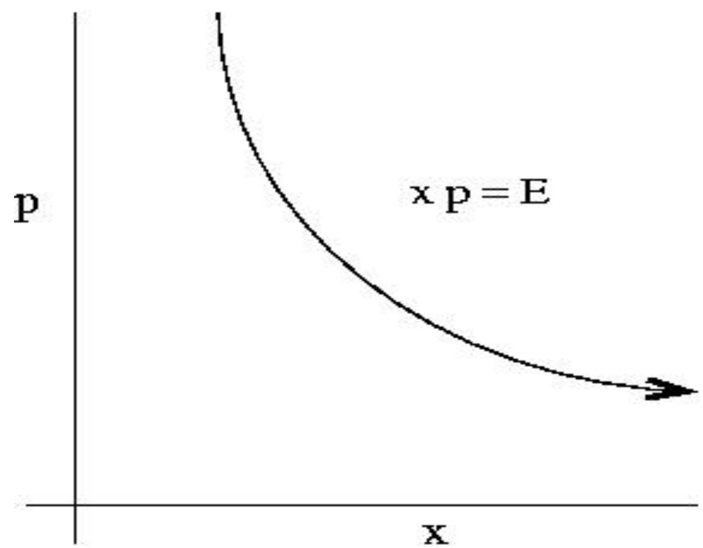
The Riemann Hamiltonian is a quantization of

$$H = xp$$

The classical H = xp model

The classical trayectories are hyperbolae in phase space

$$x(t) = x_0\, e^{t} \;, p(t) = p_0\, e^{-t} \;, E = x_0\, p_0$$

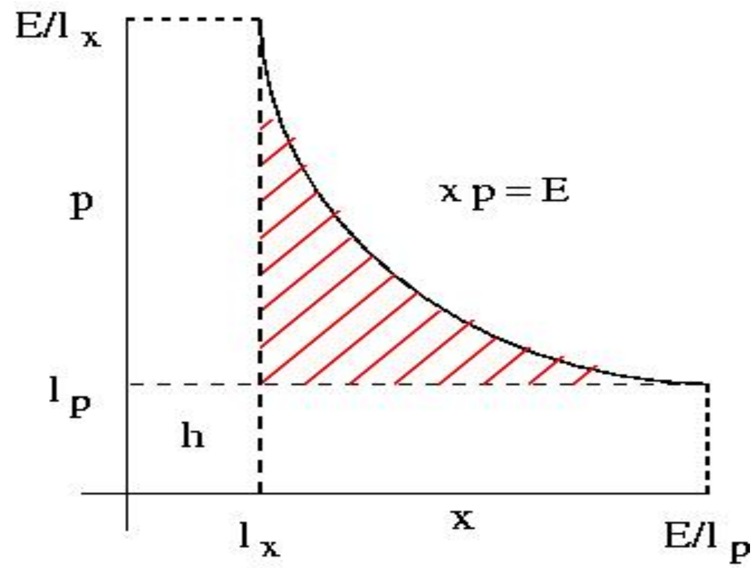


Unbounded
trayectories

Time Reversal Symmetry is broken ($x \to x,\; p \to -p$)

Berry-Keating regularization

Planck cell in phase space: $|x| > \ell_x, \quad |p| > \ell_p, \quad h = \ell_x \ell_p = 2\pi \quad (\hbar = 1)$



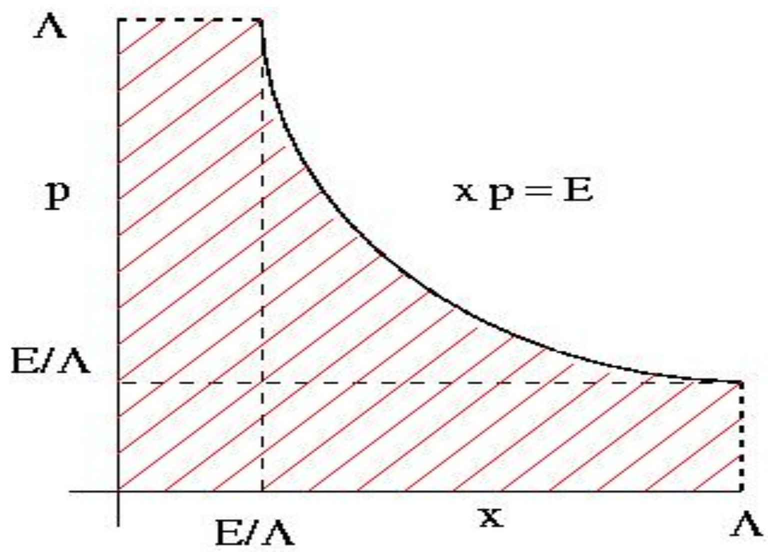
Berry−Keating cutoffs

Number of semiclassical states $N_{sc}(E) \approx \dfrac{E}{2\pi} \log \dfrac{E}{2\pi} - \dfrac{E}{2\pi} + \dfrac{7}{8}$

Agrees with number of zeros asymptotically (average term)

Connes regularization

Cutoffs in phase space: $|x| < \Lambda, \quad |p| < \Lambda$



Connes cutoffs

Number of semiclassical states $$N_{sc}(E) \approx \frac{E}{2\pi}\log\frac{\Lambda^2}{2\pi} - \frac{E}{2\pi}\log\frac{E}{2\pi} + \frac{E}{2\pi}$$

As $\Lambda \longrightarrow \infty$ spectrum = continuum - Riemann zeros

The zeros are missing spectral in a continuum

Are there quantum versions of the Berry-Keating/Connes “semiclassical” models?

OUTLINE

I) Quantize $H = xp$ $\longrightarrow$ Continuum spectrum

II) $H = x\,(p + 1/p)$ $\longrightarrow$ Discrete spectrum

III) Reformulation of $x\,(p+1/p)$ using the Dirac equation

IV) Introduce the prime numbers in an ideal model of mirrors

V) The final model

Quantization of H = xp

Define the normal ordered operator in the half-line

$$H_0 = \frac{1}{2}(x\,\hat{p} + \hat{p}\,x) = -i(x\frac{d}{dx} + \frac{1}{2})$$

$$0 < x < \infty$$

H is a self-adjoint operator: eigenfunctions

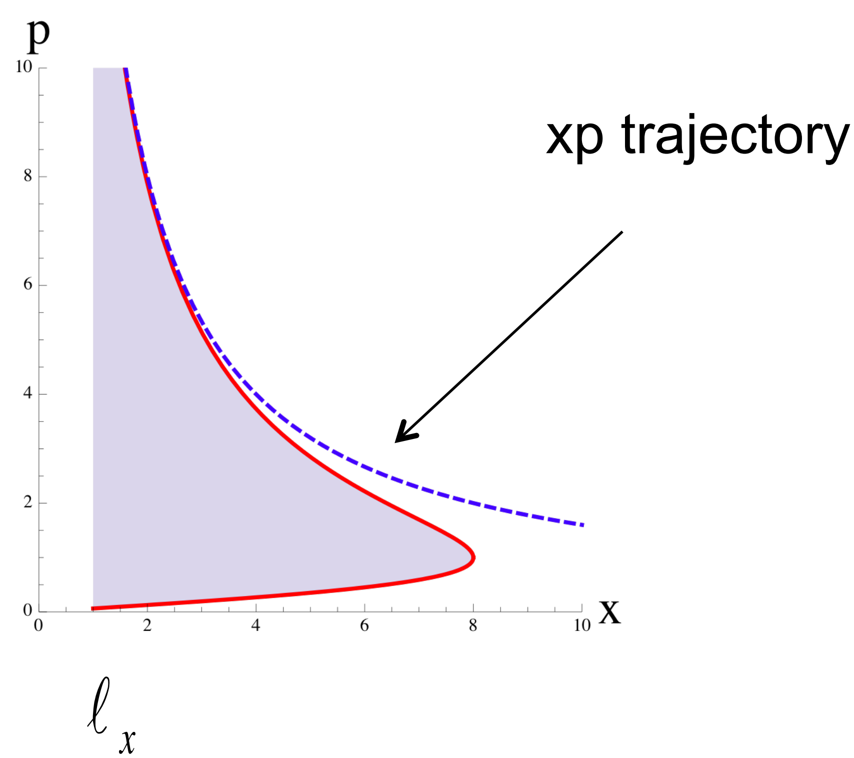
$$\phi_E(x) = \frac{1}{x^{1/2 - i\,E}}$$

The spectrum is a continuum $E \in (-\infty, \infty)$

The xp model revisited

(with J. Rodriguez-Laguna 2011)

$$H = x\left(p + \frac{\ell_p^2}{p}\right), \quad x \geq \ell_x$$



Classical trayectories are bounded and periodic

The semiclassical spectrum agrees with the smooth zeros

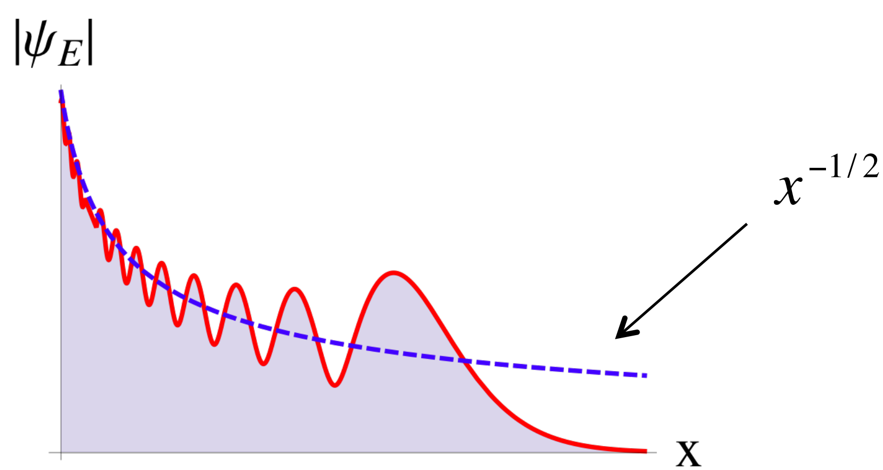
Quantization

$$H = \sqrt{x}\left(\hat{p} + \frac{\ell_p^2}{\hat{p}}\right)\sqrt{x}$$

$$\widehat{H}\psi(x) = -ix^{\frac{1}{2}}\left[\hbar\frac{d}{dx}\left\{x^{\frac{1}{2}}\psi(x)\right\} + \ell_p^2\int_{\ell_x}^{\infty}\frac{dy}{\hbar}\,\theta(y-x)y^{\frac{1}{2}}\psi(y)\right]$$

Eigenfunctions

$$\psi_E(x) = x^{\frac{iE}{2\hbar}}\,K_{\frac{1}{2}-\frac{iE}{2\hbar}}\left(\frac{\ell_p x}{\hbar}\right) \sim \begin{cases} x^{-\frac{1}{2}+\frac{iE}{\hbar}} & x << x_{\mathrm{m}}(E) \\ x^{-\frac{1}{2}+\frac{iE}{2\hbar}}\,e^{-\ell_p x/\hbar} & x >> x_{\mathrm{m}}(E) \end{cases}$$



H is selfadjoint acting on the wave functions satisfying

$$\hbar \ell_x^{\frac{1}{2}} e^{i\vartheta}\, \psi(\ell_x) + \ell_p \int_{\ell_x}^{\infty} dx\, x^{\frac{1}{2}} \psi(x) = 0.$$

Which yields the eq. for the eigenvalues

$$K_{\frac{1}{2}+\frac{iE_n}{2\hbar}}\left(\frac{\ell_x \ell_p}{\hbar}\right) - e^{i\vartheta} K_{\frac{1}{2}-\frac{iE_n}{2\hbar}}\left(\frac{\ell_x \ell_p}{\hbar}\right) = 0.$$

ϑ parameter of the self-adjoint extension

$\vartheta = 0 \leftrightarrow E_n, -E_n, \quad E_0 = 0$ Periodic

$\vartheta = \pi \leftrightarrow E_n, -E_n, \quad E_0 \neq 0$ Antiperiodic

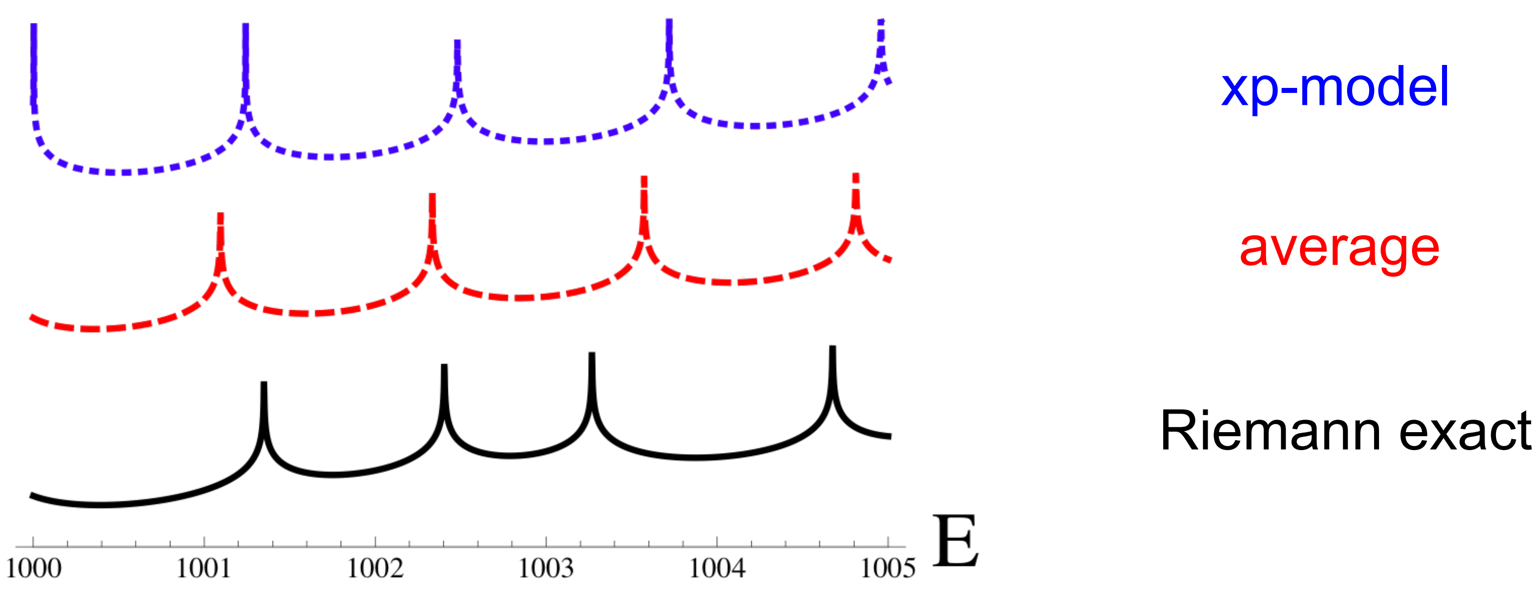
Riemann zeros also appear in pairs and 0 is not a ***zero***, i.e $\varsigma(1/2) \neq 0$

$\longrightarrow \vartheta = \pi$

In the limit $E/\ell_x \ell_p >> 1$ $\ell_x \ell_p = 2\pi$

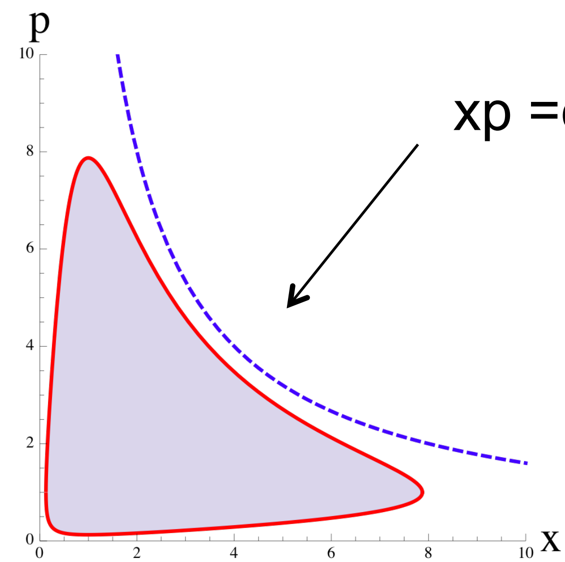
$$n(E) \approx \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right) + \frac{1}{2} + O(1/E)$$

Agrees with first two terms in Riemann formula



Berry-Keating modification of xp (2011)

$$H = \left(x + \frac{\ell_x^2}{x}\right)\left(p + \frac{\ell_p^2}{p}\right), \quad x \geq 0$$



$$\hat{H} = \left(x + \frac{\ell_x^2}{x}\right)^{1/2}\left(\hat{p} + \frac{\ell_p^2}{\hat{p}}\right)\left(x + \frac{\ell_x^2}{x}\right)^{1/2}$$

$$N_{t,\alpha=0}(t) = \frac{t}{2\pi}\left(\log\left(\frac{t}{2\pi}\right) - 1\right) - \frac{8\pi}{t}\log\left(\frac{t}{2\pi}\right) + \cdots$$

Same as Riemann but the 7/8 is also missing

$$H = \sqrt{x}\left(\hat{p} + \frac{\ell_p^2}{\hat{p}}\right)\sqrt{x}$$

$$\hat{H} = \left(x + \frac{\ell_x^2}{x}\right)^{1/2}\left(\hat{p} + \frac{\ell_p^2}{\hat{p}}\right)\left(x + \frac{\ell_x^2}{x}\right)^{1/2}$$

Average Riemann zeros

The fluctuations are NOT explained

Reason: they are non chaotic Hamiltonians. All orbits are periodic.
No relation with prime numbers.

Moreover the system must be quasi-one dimensional

Egyptian Paradox: how to get chaos in a 1D billiard?

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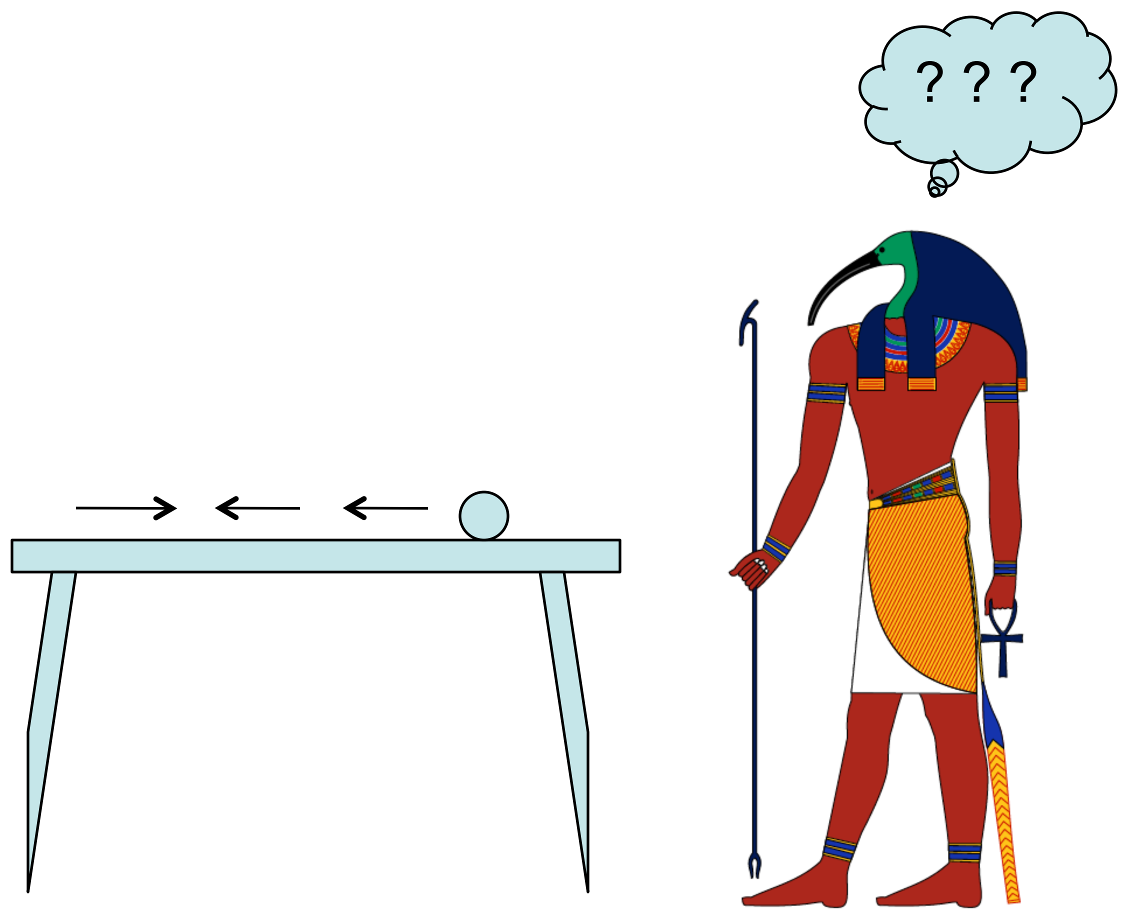
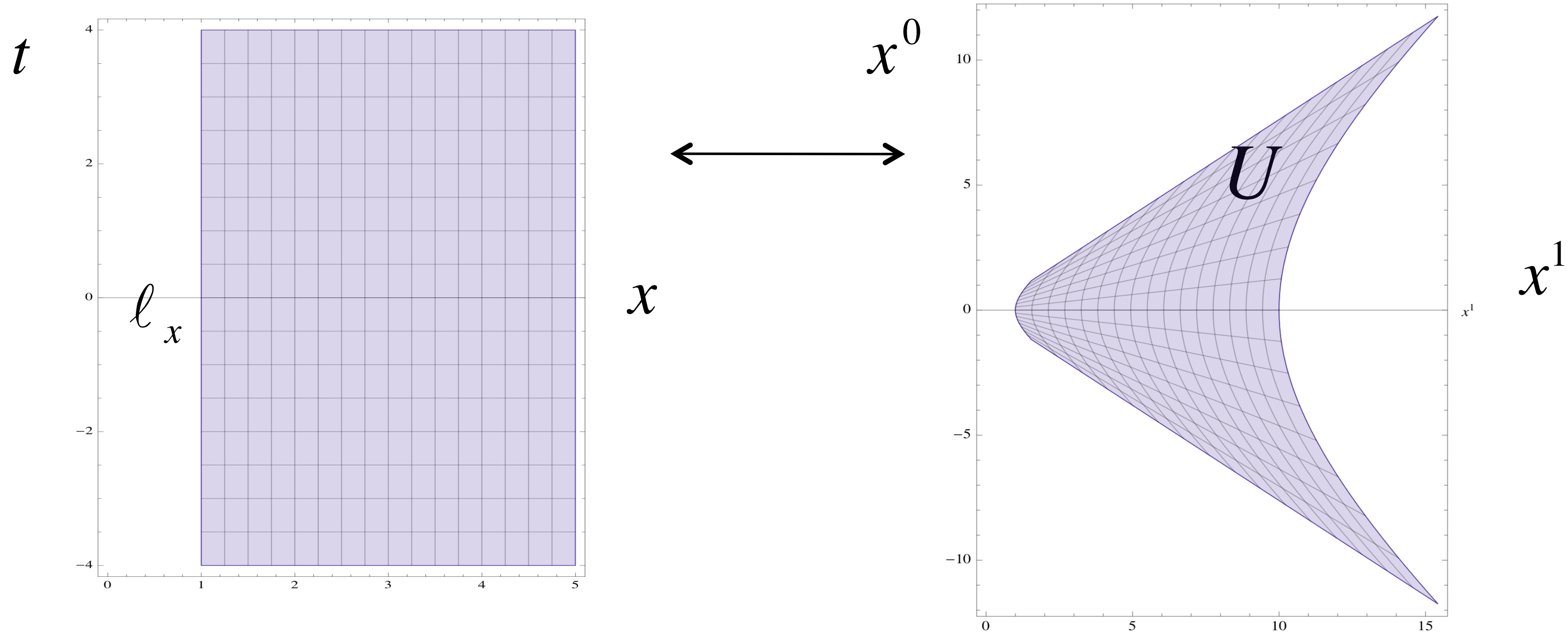


The x(p+1/p) model
and
the Dirac equation

ℓ_x

Change of variables $$t = \frac{1}{2}\log(x^0 + x^1), \quad x = \sqrt{(x^1)^2 - (x^0)^2}$$



Rindler coordinates $$x^0 = \rho \sinh\phi, \; x^1 = \rho \cosh\phi$$

Minkowski metric $$ds^2 = d\rho^2 - \rho^2 \, d\phi^2$$

Domain U: $$\rho \geq \ell_x$$

Special relativity

$\rho = \ell = cte$ = worldline of an observer moving with

uniform acceleration = $1/\ell$

ϕ = Rindler time

The domain U is invariant under the shifts of ϕ

$\partial/\partial\phi$ Killing vector of the metric

Dirac fermion in Rindler space (GS 2014)

$$S = \frac{i}{2} \int_U d^2 x \, \overline{\psi} \left(\gamma^{\mu} \partial_{\mu} + m \right) \psi$$

Dirac equation $\quad i \partial_{\phi} \chi = H_R \chi, \quad \chi = \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix}$

Rindler Hamiltonian $\quad H_R = \begin{pmatrix} -i(\rho \partial_{\rho} + 1/2) & m \rho \\ m \rho & i(\rho \partial_{\rho} + 1/2) \end{pmatrix}$

Two copies of the quantum xp one for each chirality

Boundary condition $\quad -i e^{i\vartheta} \chi_- = \chi_+ \quad at \quad \rho = \ell_x$

The eigenfunctions of H_R

$$\chi_{\pm}(\rho,\phi) = e^{-iE\phi \mp i\pi/4} K_{1/2 \pm iE}(m\rho)$$

The boundary condition gives the eigenvalue equation

$$e^{i\vartheta} K_{1/2-iE}(m\ell_x) - K_{1/2+iE}(m\ell_x) = 0$$

We recover the spectrum of the x(p+1/p) model

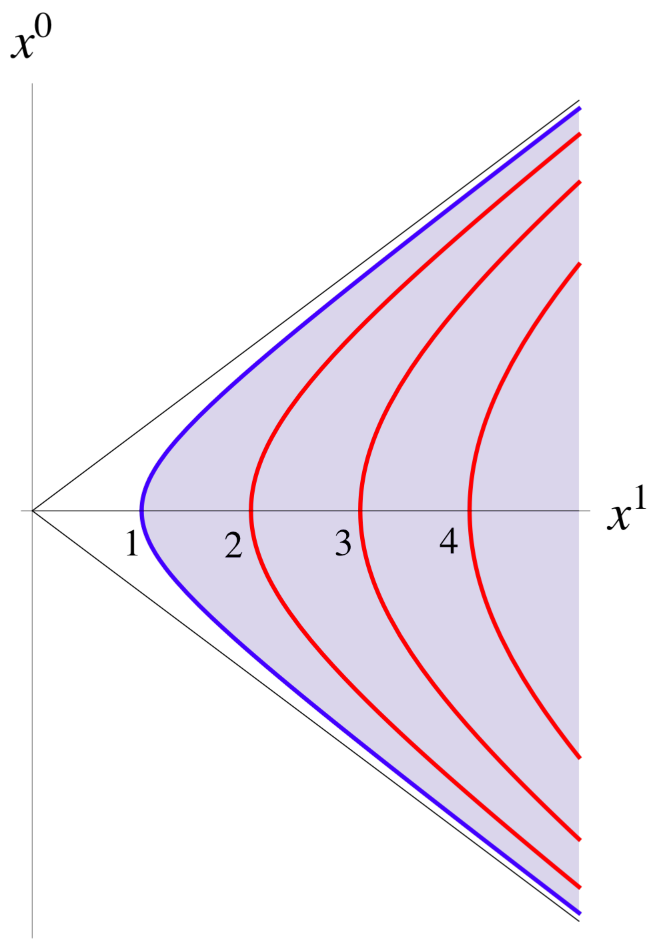
ℓ_p the mass and $1/\ell_x$ the acceleration of the boundary

How to include the fluctuations?

Resolution of the “egyptian” paradox

An ideal optical model

Moving mirror: is a mirror whose acceleration is constant $a_n = 1/\ell_n$



First mirror is perfect (n=1)

Mirrors n=2,3,…. Are semitransparent (beam splitters)

$$\ell_n = \ell_1 \sqrt{n}$$

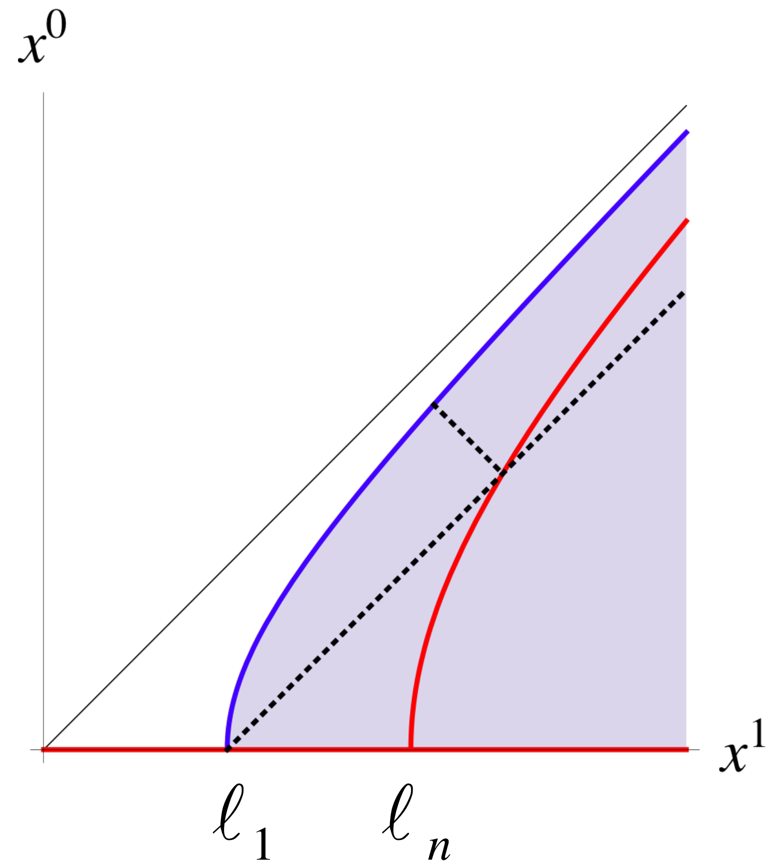
Light ray trayectory: A $\longrightarrow$ B

$$(\rho_A, \phi_A) \to (\rho_B, \phi_B) \qquad d\rho = \pm \rho\, d\phi \to \phi_B - \phi_A = \left| \log \frac{\rho_B}{\rho_A} \right|$$

Periodic trajectory:

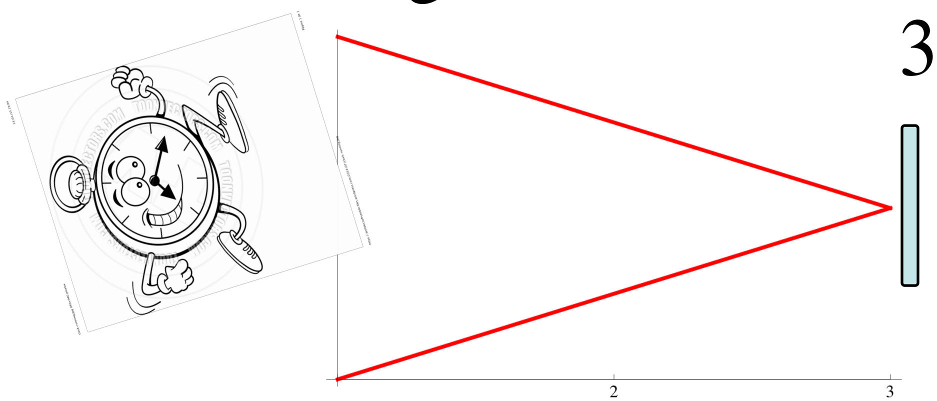
$$\ell_1 \longrightarrow \ell_n \longrightarrow \ell_1$$

$$\tau_n = 2 \log \frac{\ell_n}{\ell_1} = \log n$$



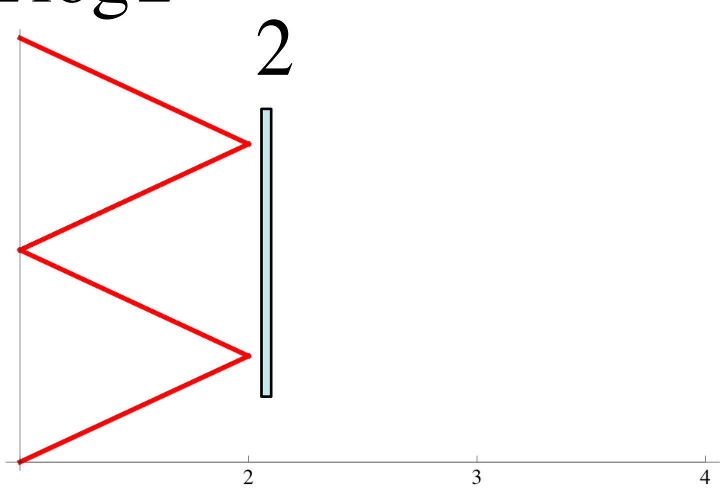
prime numbers correspond to unique paths

$$\tau = \log 3$$

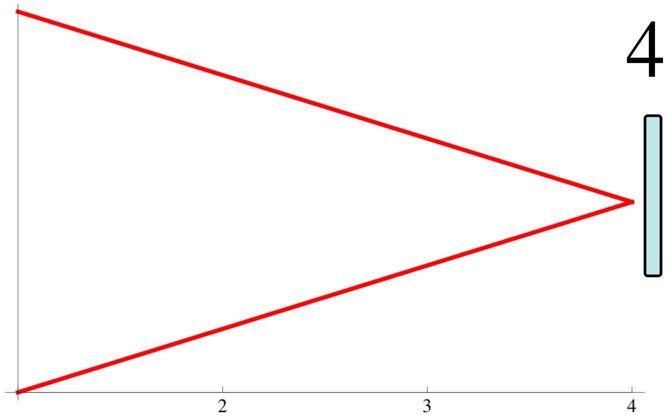


composite numbers correspond to different paths with same period

$$\tau = 2\log 2$$



$$\tau = \log 4$$



Geometrical Optics $\Rightarrow$ Prime numbers

Geometrical Optics $\Longrightarrow$ Prime numbers

Physical Optics $\Longrightarrow$ Riemann zeros

Interference

The final model

- light-rays $\longrightarrow$ massless fermions

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- moving mirrors $\longrightarrow$ delta potentials at ℓ_n

reflection amplitudes r_n

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The fermion moves freely between the mirrors with Hamiltonian

$$H_R = \begin{pmatrix} -i(\rho\partial_\rho + 1/2) & 0 \\ 0 & i(\rho\partial_\rho + 1/2) \end{pmatrix}$$

The final model

- light-rays $\longrightarrow$ masless fermions
- moving mirrors $\longrightarrow$ delta potentials at ℓ_n
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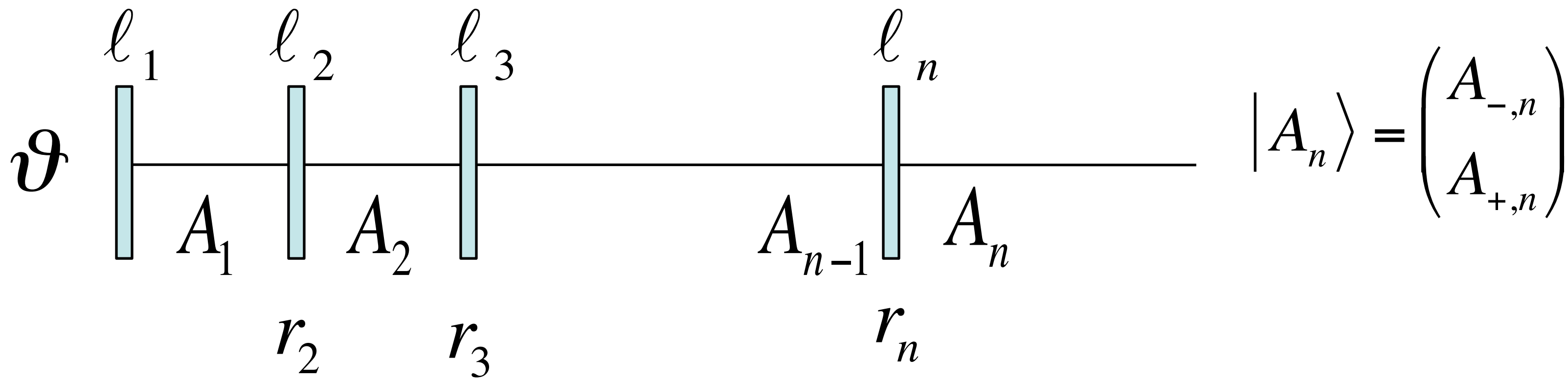
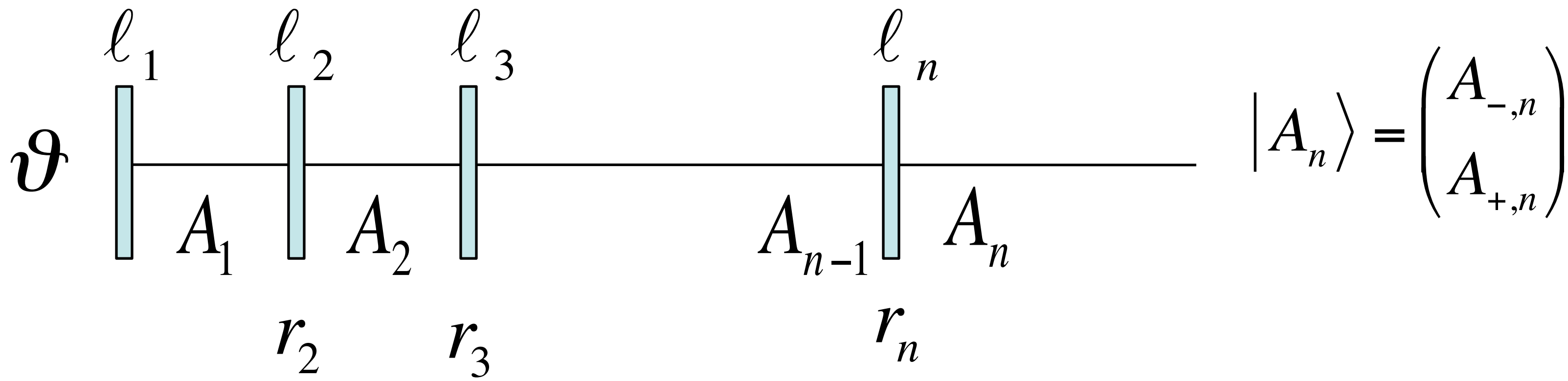
The fermion moves freely between the mirrors with Hamiltonian

$$H_R = \begin{pmatrix} -i(\rho\partial_\rho + 1/2) & 0 \\ 0 & i(\rho\partial_\rho + 1/2) \end{pmatrix}$$

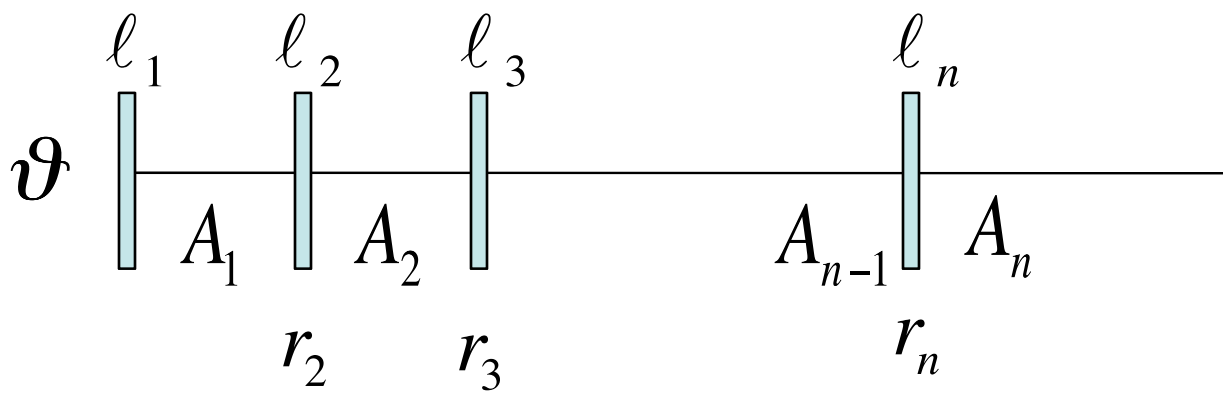
Eigenfunctions (chiral)

$$\chi_{\mp,n} \propto A_{\mp,n}\, \rho^{-1/2 \pm iE}, \quad \rho \in (\ell_n, \ell_{n+1})$$


ϑ
ℓ_1
ℓ_2
ℓ_3
ℓ_n
A_1
A_2
A_{n-1}
A_n
r_2
r_3
r_n
$|A_n\rangle = \begin{pmatrix} A_{-,n} \\ A_{+,n} \end{pmatrix}$



Boundary condition at $\ell_1 \longrightarrow |A_1\rangle = \begin{pmatrix} 1 \\ e^{i\vartheta} \end{pmatrix}$

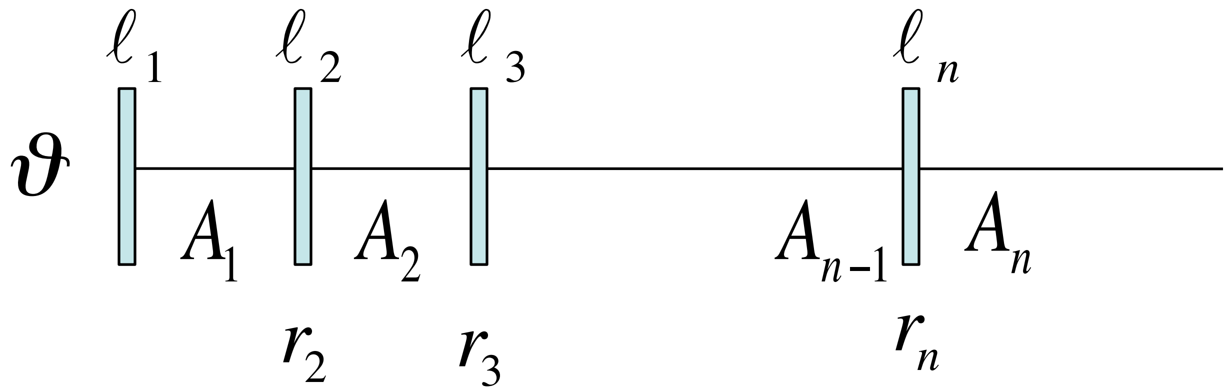


$$|A_n\rangle = \begin{pmatrix} A_{-,n} \\ A_{+,n} \end{pmatrix}$$

Boundary condition at $\ell_1 \longrightarrow |A_1\rangle = \begin{pmatrix} 1 \\ e^{i\vartheta} \end{pmatrix}$

Matching condition at $\ell_{n>1} \longrightarrow |A_{n-1}\rangle = T_n |A_n\rangle$

$$T_n = \frac{1}{1-r_n^2}\begin{pmatrix} 1+r_n^2 & 2r_n\,\ell_n^{-2iE} \\ 2r_n\,\ell_n^{2iE} & 1+r_n^2 \end{pmatrix}$$



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Norm of states $\quad \langle \chi | \chi \rangle = \int_{\ell_1}^{\infty} d\rho \left(\chi_-^* \chi_- + \chi_+^* \chi_+ \right) = \sum_{n=1}^{\infty} \log \frac{\ell_{n+1}}{\ell_n} \langle A_n | A_n \rangle$

Solution in the reflectionless limit: $r_n = O(\varepsilon) \to 0$

$$T_n = e^{\tau_n} + O(\varepsilon^2) \qquad \tau_n = \begin{pmatrix} 0 & 2r_n \ell_n^{-2iE} \\ 2r_n \ell_n^{2iE} & 0 \end{pmatrix}$$

Solution in the reflectionless limit: $r_n = O(\varepsilon) \to 0$

$$T_n = e^{\tau_n} + O(\varepsilon^2) \qquad \tau_n = \begin{pmatrix} 0 & 2r_n \ell_n^{-2iE} \\ 2r_n \ell_n^{2iE} & 0 \end{pmatrix}$$

Recursive solution

$$|A_k\rangle = e^{-\tau_k} \cdots e^{-\tau_2} |A_1\rangle \approx \exp\left(-\sum_{n=1}^{k} \tau_n\right) |A_1\rangle$$

Solution in the reflectionless limit: $r_n = O(\varepsilon) \to 0$

$$T_n = e^{\tau_n} + O(\varepsilon^2) \qquad \tau_n = \begin{pmatrix} 0 & 2r_n \ell_n^{-2iE} \\ 2r_n \ell_n^{2iE} & 0 \end{pmatrix}$$

Recursive solution

$$\left|A_k\right\rangle = e^{-\tau_k} \cdots e^{-\tau_2} \left|A_1\right\rangle \approx \exp\left(-\sum_{n=1}^{k} \tau_n\right)\left|A_1\right\rangle$$

$$\left\langle A_k \middle| A_k \right\rangle \approx e^{2R_k}\left(1-\cos\left(\Phi_k - \vartheta\right)\right) + e^{-2R_k}\left(1+\cos\left(\Phi_k - \vartheta\right)\right)$$

$$e^{-i\Phi_k} R_k = \sum_{n=1}^{k} r_n \ell_n^{-2iE}$$

The Riemann model

$$\ell_n = \sqrt{n}\,, \quad r_n = \varepsilon \frac{\mu(n)}{n^{\sigma}}$$

Moebius function

For prime numbers

$$\mu(p) = -1, \;\; \mu\left(p_1\, p_2\right) = 1, \quad \mu\left(p_1 \cdots p_r\right) = (-1)^r$$

$$\mu(p^2) = 0, \quad \mu(\cdots p^2 \cdots) = 0$$

Fermi like-statistics

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad\qquad \frac{1}{\varsigma(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

The Riemann model

$$\ell_n = \sqrt{n}\,, \quad r_n = \varepsilon \frac{\mu(n)}{n^{\sigma}}$$

$$e^{-i\Phi_\infty} R_\infty = \frac{\varepsilon}{\varsigma(\sigma + iE)}$$

The Riemann model

$$\ell_n = \sqrt{n}\,, \quad r_n = \varepsilon \frac{\mu(n)}{n^{\sigma}}$$

$$e^{-i\Phi_\infty} R_\infty = \frac{\varepsilon}{\varsigma(\sigma + iE)}$$

For $\sigma > 1/2$ the spectrum is a continuum

The Riemann model

$$\ell_n = \sqrt{n}, \quad r_n = \varepsilon \frac{\mu(n)}{n^{\sigma}}$$

$$e^{-i\Phi_\infty} R_\infty = \frac{\varepsilon}{\varsigma(\sigma + iE)}$$

For $\sigma > 1/2$ the spectrum is a continuum

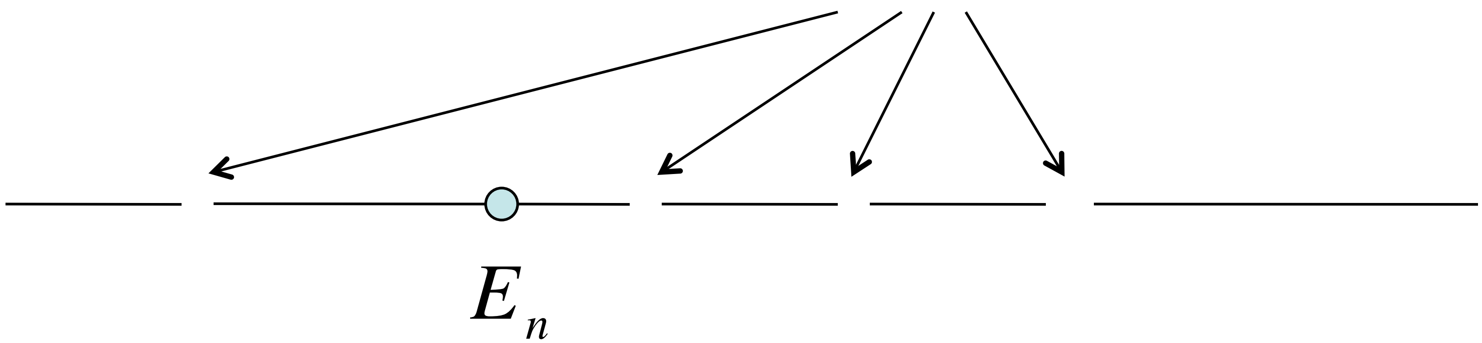
For $\sigma = 1/2$ the Riemann zeros are bound states if

$$\vartheta = -\theta(E_n) + \pi(n + 1/2)$$

$$\varsigma\left(1/2 + i\,E\right) = Z(E)\, e^{-i\theta(E)}$$

Norm of the state is $$\langle \chi | \chi \rangle \approx \varsigma \left(1 + \frac{2\varepsilon}{|Z'(E_n)|} \right)$$

For other values of ϑ the zeros are missing spectral lines



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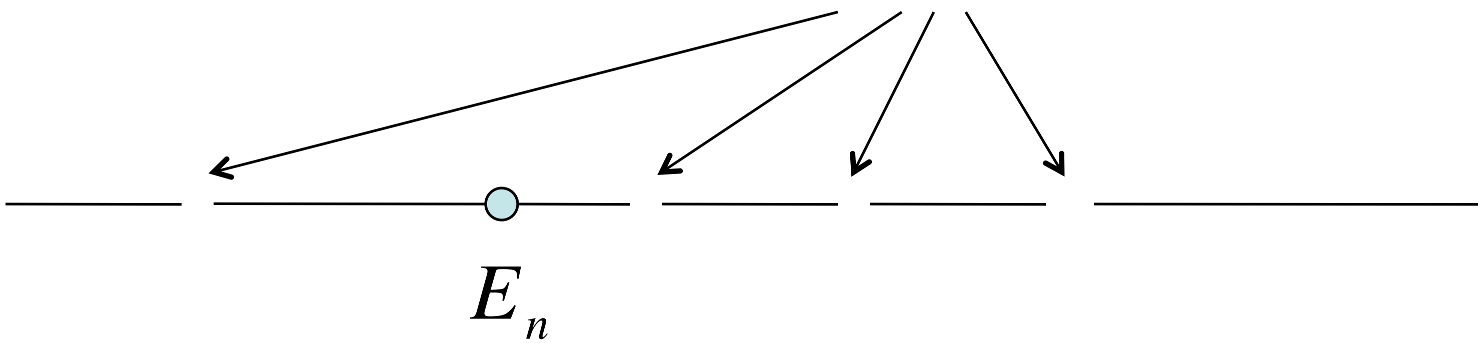

Moreover under certain assumptions one can show that there are not zeros outside the critical line

-> Proof of the Riemann hypothesis

Conclusions and further work

- The Dirac model with delta function potentials related to prime numbers provide a spectral realization of the Riemann zeros

- The realization is close to the Connes one (zeros =missing lines) but contains the zeros by fine tuning of the parameter ϑ

- Using the equivalence principle (gravity = acceleration) one can formulate the model in terms of static mirrors in a gravitational field

- The model can be generalized to Dirichlet L- functions associated to Dirichlet characters χ

$$r_n = \varepsilon \frac{\mu(n)\,\chi(n)}{n^{\sigma}}$$

- Basis for a rigourous proof of the Riemann hypothesis

Thanks for your attention