Uniform convergence of Hermite-Padé approximants for different systems of Markov type functions.

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Outline

1. Basics and recents results about convergence of Hermite-Padé approximants

2. Uniform convergence of Mixed type Hermite-Padé approximants

3. Future works
Markov’s functions

Let $s$ be a finite Borel measure with constant sign whose compact support consists of infinitely many points and is contained in the real line. By $\Delta$ we denote the smallest interval which contains the support of $s$. We denote this class of measures by $\mathcal{M}(\Delta)$. Let

$$\hat{s}(z) = \int_{\Delta} \frac{ds(x)}{z-x}$$

denote the Cauchy transform of $s$.

$\hat{s}(z)$ is an analytic function in any domain $\Omega \subset \bar{\mathbb{C}} \setminus \Delta$. 
**Padé approximants**

**Definition (Padé approximants)**

*Fix a non zero number* \( n \in \mathbb{N} \), *then there exists a polynomials* \( P_n \) *and* \( Q_n \) *such that*

1. \( \deg P_n \leq n - 1, \deg Q_n \leq n \quad Q_n \neq 0 \)
2. \( Q_n(z)\hat{s}(z) - P_n(z) = O(1/z^{n+1}), \quad z \to \infty, \)

*The unique rational function* \( \frac{P_n}{Q_n} \) *is called diagonal Padé approximants of* \( \hat{s} \).
Markov’s theorem

Theorem (A.A. Markov, 1895)

If $\Delta$ is bounded, we have

$$\lim_{n \to \infty} \frac{P_n(z)}{Q_n(z)} = \hat{s}(z), \quad K \subset \overline{\mathbb{C}} \setminus \Delta.$$
Let $\Delta_\alpha$ and $\Delta_\beta$ be two non intersecting bounded intervals contained in the real line and $\sigma_\alpha \in \mathcal{M}(\Delta_\alpha)$, $\sigma_\beta \in \mathcal{M}(\Delta_\beta)$. With these two measures we define a third one as follows

$$d\langle \sigma_\alpha, \sigma_\beta \rangle(x) = \hat{\sigma}_\beta(x)d\sigma_\alpha(x).$$
**Definition (Nikishin system)**

Take a collection $\Delta_j, j = 1, \ldots, m$, of intervals such that

$$\Delta_j \cap \Delta_{j+1} = \emptyset, \quad 1, \ldots, m-1.$$  

Let $(\sigma_1, \ldots, \sigma_m)$ be a system of measures such that $\text{Co}(\text{supp}(\sigma_j)) = \Delta_j,$  
$\sigma_j \in \mathcal{M}(\Delta_j) \quad j = 1, \ldots m.$  
We say that $(s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$, where

$$s_{1,1} = \sigma_1, \quad s_{1,2} = \langle \sigma_1, \sigma_2 \rangle, \ldots, s_{1,m} = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_m \rangle \rangle$$

is the Nikishin system of measures generated by $(\sigma_1, \ldots, \sigma_m)$.

Take $j \leq k$ we denote

$$s_{j,k} = \langle \sigma_j, \sigma_{j+1}, \ldots, \sigma_k \rangle, \quad s_{j,j} = \langle \sigma_j \rangle = \sigma_j, \quad s_{k,j} = \langle \sigma_k, \sigma_{k-1}, \ldots, \sigma_j \rangle$$
Type II Hermite-Padé approximants

**Definition (Type II Hermite-Padé approximants)**

Let \( s(z) = (s_{1,1}, \ldots, s_{1,m}) \) be Nikishin system. Fix a non zero multi-index \( \vec{n} = (n_1, \ldots, n_m) \in \mathbb{N}_{+}^m, |\vec{n}| = n_1 + \ldots + n_m \). There exists a polynomial \( Q_{\vec{n}} \) such that

\[
\begin{align*}
&i) \quad \text{deg } Q_{\vec{n}} \leq |\vec{n}| \quad Q_{\vec{n}} \neq 0 \\
&ii) \quad Q_{\vec{n}}(z)\hat{s}_{1,j}(z) - P_{\vec{n},j}(z) = O(1/z^{n_j+1}), \quad z \to \infty, \quad j = 1, \ldots, m
\end{align*}
\]

for some polynomials \((P_{\vec{n},1}, \ldots, P_{\vec{n},m})\). The vector of rational functions \((\frac{P_{\vec{n},1}}{Q_{\vec{n}}}, \ldots, \frac{P_{\vec{n},m}}{Q_{\vec{n}}})\) is called type II Hermite-Padé approximants of \( \hat{s} \) respect to the multi-index \( \vec{n} \).

If \( m = 1 \), \( \frac{P_{\vec{n},1}}{Q_{\vec{n}}} \) is the classical Padé approximants of \( \hat{s}_{1,1} \).
Analog of Markov’s Theorem for type II

**Theorem**

Let \( s(z) = (s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m) \) be a Nikishin system, \( \Lambda \subset \mathbb{Z}_+^m \) satisfies

\[
  n_1 \geq \ldots \geq n_m \geq n_1 - c, \quad j = 1, \ldots, m, \quad \vec{n} \in \Lambda.
\]

Then, for \( j = 1, \ldots, m \)

\[
  \lim_{|\vec{n}| \to \infty} \frac{P_{\vec{n},j}(z)}{Q_{\vec{n}}(z)} = \hat{s}_{1,j}(z), \quad \mathcal{K} \subset \overline{\mathbb{C}} \setminus \Delta.
\]

Type I Hermite-Padé approximants

**Definition (Type I Hermite-Padé approximants)**

Let \( s(z) = (s_{1,1}, \ldots, s_{1,m}) \) be a Nikishin system. Fix a non zero multi-index \( \vec{n} = (n_0, \ldots, n_m) \in \mathbb{N}_+^{m+1} \), where \( n_0 - 1 \geq n_j = 1, \ldots, m \). There exist polynomials \( a_{\vec{n},0}, a_{\vec{n},1}, \ldots, a_{\vec{n},m} \), not all identically equal to zero, such that

\[
\begin{align*}
&i) \quad \deg a_{\vec{n},j} \leq n_j - 1, \quad j = 0, \ldots, m \\
&ii) \sum_{j=1}^{m} a_{\vec{n},j}(z)\hat{s}_{1,j}(z) + a_{\vec{n},0}(z) = O\left(1/z^{|\vec{n}|^{-n_0}}\right), \quad z \to \infty,
\end{align*}
\]

The vector of polynomials \( a_{\vec{n},0}, a_{\vec{n},1}, \ldots, a_{\vec{n},m} \) is called type I Hermite-Padé polynomials of \( \hat{s} \) respect to the multi-index \( \vec{n} \).

If \( m = 1 \), \( \frac{a_{\vec{n},0}}{a_{\vec{n},1}} \) is the classical Padé approximants of \( \hat{s}_{1,1} \).
Analog of Markov’s Theorem for type I

Theorem

Consider a sequence of multi-indices \( \vec{n} \in \Lambda \). Let \( (s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m) \) be a Nikishin system then

\[
\frac{a_{\vec{n},j}}{a_{\vec{n},m}}(z) \Rightarrow (-1)^{m-j} \hat{s}_{m,j+1}(z), \quad |\vec{n}| \to \infty \text{ on } \mathbb{C} \setminus \Delta_m \quad j = 0, \ldots, m - 1
\]

\[
s_{k,j} = \langle \sigma_k, \sigma_{k-1}, \ldots, \sigma_j \rangle \quad (s_{m,1}, \ldots, s_{m,m}) = \mathcal{N}(\sigma_m, \ldots, \sigma_1)
\]

Analog of Markov’s Theorem for type I

**Theorem**

Consider a sequence of multi-indices \( \vec{n} \in \Lambda \). Let \((s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)\) be a Nikishin system then

\[
\frac{a_{\vec{n},j}}{a_{\vec{n},m}}(z) \Rightarrow (-1)^{m-j} \hat{s}_{m,j+1}(z) \quad , |\vec{n}| \to \infty \text{ on } \mathbb{C} \setminus \Delta_m \quad j = 0, \ldots m - 1
\]

\[
s_{k,j} = \langle \sigma_k, \sigma_{k-1}, \ldots, \sigma_j \rangle \quad (s_{m,1}, \ldots, s_{m,m}) = \mathcal{N}(\sigma_m, \ldots, \sigma_1)
\]

Extensions of Markov’s theorem

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Type I Hermite-Padé approximants

Let \((r_1 = \frac{v_1}{t_1}, \ldots, r_m = \frac{v_m}{t_m})\) be real rational functions where the poles of \(r_j\) lie in \(\mathbb{C} \setminus (\Delta_1 \cup \Delta_m)\) and \(\deg t_j = d_j; \deg v_j < d_j\), for all \(j = 1, \ldots, m\).

Definition (Type I Hermite-Padé approximants)

Let \(s(z) = (s_{1,1}, \ldots, s_{1,m})\) be a Nikishin system. Fix a non zero multi-index \(\vec{n} = (n_0, \ldots, n_m) \in \mathbb{N}_+^{m+1}\), where \(n_0 - 1 \geq n_j = 1, \ldots m\). There exist polynomials \(a_{\vec{n},0}, a_{\vec{n},1}, \ldots, a_{\vec{n},m}\), not all identically equal to zero, such that

i) \(\deg a_{\vec{n},j} \leq n_j - 1, j = 0, \ldots, m\)

ii) \(\sum_{j=1}^m a_{\vec{n},j}(z)(\hat{s}_{1,j}(z)+r_j(z)) + a_{\vec{n},0}(z) = \mathcal{O}(1/|z|^{\vec{n}-n_0}), z \to \infty,\)

The vector of polynomials \(a_{\vec{n},0}, a_{\vec{n},1}, \ldots, a_{\vec{n},m}\) is called type I Hermite-Padé polynomials of \(\hat{s}_*\) respect to the multi-index \(\vec{n}\).
Type I Markov’s Theorem for meromorphic functions

**Theorem**

Consider a sequence of multi-indices $\vec{n} \in \Lambda$. Let $(s_{1,1}, \ldots, s_{1,m}) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ be a Nikishin system then

$$\frac{a_{\vec{n},j}}{a_{\vec{n},m}}(z) \Rightarrow (-1)^{m-j} \hat{s}_{m,j+1}(z), \quad |\vec{n}| \to \infty \text{ on } \mathbb{C} \setminus \Delta_m \quad j = 1, \ldots, m - 1$$

and

$$\lim_{n \in \Lambda} \frac{a_{n,0}}{a_{n,m}} = (-1)^m \hat{s}_{m,1} - \sum_{j=1}^{m-1} (-1)^{m-j} r_j \hat{s}_{m,j+1} + r_m.$$

Notice that the rational fractions $(r_1, \ldots, r_m)$ do not play any role in the expression of the limit of $(\frac{a_{n,1}}{a_{n,m}}, \ldots, \frac{a_{n,m-1}}{a_{n,m}})$. On the other hand, all the information of $(r_1, \ldots, r_m)$ is contained in the expression of the limit of $\frac{a_{n,0}}{a_{n,m}}$. 
Type II Hermite-Padé approximants

Let \((r_1 = \frac{v_1}{t_1}, \ldots, r_m = \frac{v_m}{t_m})\) be real rational functions where the poles of \(r_j\) lie in \(\mathbb{C} \setminus (\Delta_1 \cup \Delta_m)\) and \(\text{deg } t_j = d_j; \text{deg } v_j < d_j, \text{ for all } j = 1, \ldots, m.\)

**Definition (Type II Hermite-Padé approximants)**

Let \(s(z) = (s_{1,1}, \ldots, s_{1,m})\) be Nikishin system. Fix a non zero multi-index \(\vec{n} = (n_1, \ldots, n_m) \in \mathbb{N}_+^m, |\vec{n}| = n_1 + \ldots + n_m.\) There exists a polynomial \(Q_{\vec{n}}\) such that

1) \(\text{deg } Q_{\vec{n}} \leq |\vec{n}| \quad Q_{\vec{n}} \not\equiv 0\)

2) \(Q_{\vec{n}}(z)(\hat{s}_{1,j}(z) + r_j) - P_{\vec{n},j}(z) = O(1/z^{n_j+1}), \quad z \to \infty, \quad j = 1, \ldots, m\)

for some polynomials \((P_{\vec{n},1}, \ldots, P_{\vec{n},m}).\) The vector of rational functions \((\frac{P_{\vec{n},1}}{Q_{\vec{n}}}, \ldots, \frac{P_{\vec{n},m}}{Q_{\vec{n}}})\) is called type II Hermite-Padé approximants of \(\hat{s}_*\) respect to the multi-index \(\vec{n}.\)
Type II Markov’s Theorem for meromorphics functions

Theorem

Fix a compact $K \subset \mathbb{C} \setminus \Delta_1$. Assume that $r_1, r_2, \ldots r_m$ have no common finite poles, and all of them lie in $\mathbb{C} \setminus (\Delta_1 \cup \Delta_m)$ then

$$\frac{P_{\vec{n},j}}{Q_{\vec{n}}} \Rightarrow \hat{s}_j + r_j, \ |n| \to \infty \text{ on } K \subset \Omega'$$

where $\Omega' = \mathbb{C} \setminus (\Delta_1 \cup \{z : \exists j = 1, \ldots, m : t_j(z) = 0\})$
Outline

1. Basics and recents results about convergence of Hermite-Padé approximants
2. Uniform convergence of Mixed type Hermite-Padé approximants
3. Future works
## Mixed type Hermite-Padé problem

### Definition

Let $n \geq 1$. Given two Nikishin system $(s_{1,1}, s_{1,2}) = \mathcal{N}(\sigma_1, \sigma_2)$, and $(s_{2,2}, s_{2,1}) = \mathcal{N}(\sigma_2, \sigma_1)$. We seek for polynomials $(a_{n,0}, a_{n,1}, a_{n,2})$, $\deg a_{n,0} \leq n - 1, \deg a_{n,1} \leq n - 1$ and $\deg a_{n,2} \leq n$ which satisfy a mixed type Hermite-Padé approximations conditions as $z \to \infty$

$$a_{n,2}(z)\hat{s}_{2,2}(z) - a_{n,1}(z) = O\left(\frac{1}{z}\right)$$  \hspace{1cm} (1) \hspace{1cm} 

$$a_{n,0}(z) - a_{n,1}\hat{s}_{1,1}(z) + a_{n,2}(z)\hat{s}_{1,2}(z) = O\left(\frac{1}{z^{n+1}}\right)$$  \hspace{1cm} (3)
### Mixed type Hermite-Padé problem

**Definition**

Let \( n \geq 1 \). Given two Nikishin system \((s_{1,1}, s_{1,2}) = \mathcal{N}(\sigma_1, \sigma_2)\), and \((s_{2,2}, s_{2,1}) = \mathcal{N}(\sigma_2, \sigma_1)\). We seek for polynomials \((a_{n,0}, a_{n,1}, a_{n,2})\), \(\deg a_{n,0} \leq n - 1, \deg a_{n,1} \leq n - 1\) and \(\deg a_{n,2} \leq n\) which satisfy a mixed type Hermite-Padé approximations conditions as \(z \to \infty\):

\[
a_{n,2}(z)\hat{s}_{2,2}(z) - a_{n,1}(z) = \mathcal{O}\left(1/z\right) \quad (1)
\]

\[
a_{n,0}(z) - a_{n,1}\hat{s}_{1,1}(z) + a_{n,2}(z)\hat{s}_{1,2}(z) = \mathcal{O}\left(1/z^{n+1}\right) \quad (3)
\]

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Mixed type Hermite-Padé problem

Definition

Let \( n \geq 1 \). Given two Nikishin system \((s_{1,1}, s_{1,2}) = N(\sigma_1, \sigma_2)\), and \((s_{2,2}, s_{2,1}) = N(\sigma_2, \sigma_1)\). We seek for polynomials \((a_{n,0}, a_{n,1}, a_{n,2})\), \(\deg a_{n,0} \leq n - 1, \deg a_{n,1} \leq n - 1\) and \(\deg a_{n,2} \leq n\) which satisfy a mixed type Hermite-Padé approximations conditions as \( z \to \infty \)

\[
a_{n,2}(z)\hat{s}_{2,2}(z) - a_{n,1}(z) = \mathcal{O}(1/z) \tag{1}
\]

\[
a_{n,2}(z)\hat{s}_{2,1}(z) - a_{n,0}(z) = \mathcal{O}(1/z) \tag{2}
\]

\[
a_{n,0}(z) - a_{n,1}\hat{s}_{1,1}(z) + a_{n,2}(z)\hat{s}_{1,2}(z) = \mathcal{O}(1/z^{n+1}) \tag{3}
\]

Theorem

Consider a sequence of vector polynomials \((a_{n,0}, a_{n,1}, a_{n,2})\) that satisfies the previous mixed type approximation problems then

\[
\frac{a_{n,1}}{a_{n,2}}(z) \Rightarrow \hat{s}_{2,2}(z) \quad , \quad n \to \infty \quad on \quad \mathbb{C} \setminus \Delta_2
\]

\[
\frac{a_{n,0}}{a_{n,2}}(z) \Rightarrow \hat{s}_{2,1}(z) \quad , \quad n \to \infty \quad on \quad \mathbb{C} \setminus \Delta_2
\]
An idea of the proof.

- \[ a_{n,0}(z) - a_{n,1}\hat{s}_{1,1}(z) + a_{n,2}(z)\hat{s}_{1,2}(z) = \mathcal{O}(1/z^{n+1}) \]
- \[ \int_{\Delta_1} x^\nu (a_{n,2}(x)\hat{s}_{2,2}(x) - a_{n,1}(x))ds_{1,1}(x) = 0, \ \nu = 0, n - 1 \]
- \[(a_{n,2}(x)\hat{s}_{2,2}(x) - a_{n,1}(x)) \text{ has at least } n \text{ simples zeros on } \Delta_1 \]
- \[ \frac{a_{n,2}(z)\hat{s}_{2,2}(z) - a_{n,1}(z)}{w_n(z)} = \mathcal{O}(1/z^{n+1}), z \to \infty, \]
- \[ a_{n,1}/a_{n,2} \text{ is a multipoint Padé approximant of } \hat{s}_{2,2} \]


- Using some properties of Nikishin system and the previous ideas we obtain the convergence of \[ a_{n,0}/a_{n,2} \] to \[ \hat{s}_{2,1} \]
Degasperis-Procesi equations

\[ u_t - u_{xxt} + 4uu_x = 3u_xu_{xx} + uu_{xxx}, (x,t) \in \mathbb{R}^2 \]

This equations admit (in a weak sense) a type of nonsmooth solutions called multipeakons (peakon = peaked soliton). These take the form of a train of peak-shaped interacting waves,

\[ u(x, t) = \sum_{i=1}^{\infty} m_i(t) e^{-|x-x_i(t)|} \]

\[ u(x, t) = \sum_{k=1}^{n} m_k(t) e^{-|x-x_k(t)|} \]
Discrete cubic string problem

Given a function $g(y) \geq 0$, determine the eigenvalues $z$ such that nontrivial eigenfunctions $\phi(y)$ exist, satisfying

$$-\phi_{yyy}(y) = zg(y)\phi(y), \quad \text{for } y \in (-1, 1)$$

$$\phi(-1) = \phi_y(-1) = 0, \quad \phi(1) = 0.$$
Mixed type Hermite-Padé problem

Definition

Let \( n \geq 1 \). Given two Nikishin system \( (s_{1,1}, s_{1,2}) = \mathcal{N}(\sigma_1, \sigma_2) \), and \( (s_{2,2}, s_{2,1}) = \mathcal{N}(\sigma_2, \sigma_1) \). We seek for polynomials \( (a_{n,0}, a_{n,1}, a_{n,2}) \), \( \deg a_{n,0} \leq n - 1, \deg a_{n,1} \leq n - 1 \) and \( \deg a_{n,2} \leq n \) which satisfy a mixed type Hermite-Padé approximations conditions as \( z \to \infty \)

\[
\begin{align*}
    a_{n,2}(z)\hat{s}_{2,2}(z) - a_{n,1}(z) &= \mathcal{O}\left(\frac{1}{z}\right) \\
    a_{n,2}(z)\hat{s}_{2,1}(z) - a_{n,0}(z) &= \mathcal{O}\left(\frac{1}{z}\right) \\
    a_{n,0}(z) - a_{n,1}\hat{s}_{1,1}(z) + a_{n,2}(z)\hat{s}_{1,2}(z) &= \mathcal{O}\left(\frac{1}{z^{n+1}}\right)
\end{align*}
\]

Outline

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2. Uniform convergence of Mixed type Hermite-Padé approximants

3. Future works
The mixed type Hermite Padé approximation problem can be extended in the sense of consider a Nikishin system of $m$ measures. Moreover, the results about uniform convergence can be obtained consider not only the case of discrete measures.

Problem

How this extension of the approximation problems and the results of the uniform convergence when the measures are not discrete are related with the inverse spectral problem for the string and the Degasperis-Procesi equation?
For the Degasperis-Procesi equation a more general case deals with peakons which move to right and atipeakons which to left. In this situation the approximation problem which arises from the Degasperis-Procesi equation is with respect to a rational pertubation of the Nikishin system.

\[
    u(x, t) = \sum_{k=1}^{n} \left( m_k(t) - s_k(t) \text{sgn}(x - x_k(t)) \right) e^{-|x-x_k(t)|}
\]

**Problem**

Can we prove an explicit peakon-antipeakon formula in this more general case?
THANK YOU!!!