Multivariate Orthogonal Polynomials and Integrable Systems

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Outline I

1. Cholesky description of multivariate orthogonal polynomials
   - Monomials
   - The moment matrix
   - Quasi-determinantal formulæ
   - MVOPR
   - Shift matrices
   - Jacobi matrices
   - Three term relations
   - Christoffel–Darboux formulæ

2. Discrete Toda flows
   - The flows
   - Translations and adjoint lattice resolvents
   - Wave matrices and lattice resolvents
   - The partial difference equations
   - Elements of integrability
   - Quasi-tau functions formulæ for MOVPR

Outline II

3. Darboux transformations
   - 1D case
   - Multivariate degree one Darboux transformation
   - The general case: degree m polynomial Darboux transformations

4. Continuous Toda flows
   - The continuous flows
   - Baker functions
   - Integrable hierarchy
   - Bilinear equations
   - Toda equations
   - KP style

Our inspiration

The research of Mark Adler and Pierre van Moerbeke on orthogonal polynomials and integrable systems

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Where

Gerardo Araznibarreta and MM, *Multivariate orthogonal polynomials and integrable systems*, `arXiv:1409.0570 [math.CA]`

Our previous work in this direction

- Carlos Álvarez-Fernández, Ulises Fidalgo Prieto, and MM, *Multiple orthogonal polynomials of mixed type: Gauss–Borel factorization and the multi-component 2D Toda hierarchy*, Advances in Mathematics **227** (2011) 1451-1525
- Carlos Álvarez-Fernández and MM, *Orthogonal Laurent polynomials on the unit circle, extended CMV ordering and 2D Toda type integrable hierarchies*, Advances in Mathematics **240** (2013) 132-193
- Gerardo Ariznabarreta and MM, *Matrix Orthogonal Laurent Polynomials on the Unit Circle and Toda Type Integrable Systems*, Advances in Mathematics **264** (2014) 396-463

- Carlos Álvarez-Fernández, Ulises Fidalgo Prieto, and MM, *The multicomponent 2D Toda hierarchy: generalized matrix orthogonal polynomials, multiple orthogonal polynomials and Riemann-Hilbert problems*, Inverse Problems **26** (2010) 055009 (15pp)
- Carlos Álvarez-Fernández and MM, *On the Christoffel–Darboux formula for generalized matrix orthogonal polynomials*, Journal of Mathematical Analysis and Applications **418** (2014) 238-247
- Gerardo Ariznabarreta and MM, *A Jacobi type Christoffel–Darboux formula for multiple orthogonal polynomials of mixed type*, Linear Algebra and its Applications **468** (2015), 154-170.

Multivariate measures and monomials

- D independent real variables $\boldsymbol{x} = (x_1, x_2, \dots, x_D)^\top \in \Omega \subseteq \mathbb{R}^D$ varying in the domain Ω, a Borel measure $\mathrm{d}\mu(x) \in \mathfrak{B}(\Omega)$ and an inner product $\langle f, g\rangle := \int_\Omega f(\boldsymbol{x}) g(\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{x})$

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- Multi-index $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_D)^\top \in \mathbb{Z}_+^D$ of non-negative integers with length $|\boldsymbol{\alpha}| := \sum_{a=1}^D \alpha_a$. We write $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_D^{\alpha_D}$ and order the monomials, $\boldsymbol{x}^{\boldsymbol{\alpha}} < \boldsymbol{x}^{\boldsymbol{\alpha}'} \Leftrightarrow |\boldsymbol{\alpha}| < |\boldsymbol{\alpha}'|$

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- For $k \in \mathbb{Z}_+$ introduce $[k] := \{\boldsymbol{\alpha} \in \mathbb{Z}_+^D : |\boldsymbol{\alpha}| = k\}$ and use the graded lexicographic order and write

$$[k] = \{\boldsymbol{\alpha}_1^{(k)}, \boldsymbol{\alpha}_2^{(k)}, \dots, \boldsymbol{\alpha}_{|[k]|}^{(k)}\} \text{ with } \boldsymbol{\alpha}_a^{(k)} < \boldsymbol{\alpha}_{a+1}^{(k)}.$$

Here $|[k]| = \left(\!\binom{D}{k}\!\right) = \binom{D+k-1}{k}$ is the cardinality of the set $[k]$, i.e., the number of elements in the set

- Vectors of monomials

$$\chi := \begin{pmatrix} \chi_{[0]} \\ \chi_{[1]} \\ \vdots \\ \chi_{[k]} \\ \vdots \end{pmatrix} \qquad \text{where} \qquad \chi_{[k]} := \begin{pmatrix} \boldsymbol{x}^{\boldsymbol{\alpha}_1} \\ \boldsymbol{x}^{\boldsymbol{\alpha}_2} \\ \vdots \\ \boldsymbol{x}^{\boldsymbol{\alpha}_{|[k]|}} \end{pmatrix}$$

$$\chi^* := \Big(\prod_{a=1}^{D} x_a^{-1}\Big)\chi(x_1^{-1},\dots,x_D^{-1})$$

- For example

$$\chi_{[0]} = 1, \qquad \chi_{[1]} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_D \end{pmatrix}, \qquad \chi_{[2]} = \begin{pmatrix} x_1^2 \\ x_1 x_2 \\ \vdots \\ x_1 x_D \\ x_2^2 \\ x_2 x_3 \\ \vdots \\ x_2 x_D \\ x_3^2 \\ \vdots \\ x_D^2 \end{pmatrix}$$

Partitioned matrices

A with a block structure induced by the graded lexicographic order

$$A = \begin{pmatrix} A_{[0],[0]} & A_{[0],[1]} & \cdots \\ A_{[1],[0]} & A_{[1],[1]} & \cdots \\ \vdots & \vdots & \end{pmatrix},$$

$$A_{[k],[\ell]} = \begin{pmatrix} A_{\boldsymbol{\alpha}^{(k)}_1, \boldsymbol{\alpha}^{(\ell)}_1} & \dots & A_{\boldsymbol{\alpha}^{(k)}_1, \boldsymbol{\alpha}^{(\ell)}_{|[\ell]|}} \\ \vdots & & \vdots \\ A_{\boldsymbol{\alpha}^{(k)}_{|[k]|}, \boldsymbol{\alpha}^{(\ell)}_1} & \dots & A_{\boldsymbol{\alpha}^{(k)}_{|[k]|}, \boldsymbol{\alpha}^{(\ell)}_{|[\ell]|}} \end{pmatrix} \in \mathbb{R}^{|[k]| \times |[\ell]|}$$

The moment matrix

Associated with the measure $\mathrm{d}\mu$ we have the following moment matrix

$$G := \int_\Omega \chi(\boldsymbol{x})\mathrm{d}\mu(\boldsymbol{x})\chi(\boldsymbol{x})^\top = \begin{pmatrix} G_{[0],[0]} & G_{[0],[1]} & \cdots \\ G_{[1],[0]} & G_{[1],[1]} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

with each entry being a rectangular matrix with real coefficients

$$G_{[k],[\ell]} := \int_\Omega \chi_{[k]}(\boldsymbol{x})\mathrm{d}\mu(\boldsymbol{x})(\chi_{[\ell]}(\boldsymbol{x}))^\top = \begin{pmatrix} G_{\boldsymbol{\alpha}_1^{(k)},\boldsymbol{\alpha}_1^{(\ell)}} & \dots & G_{\boldsymbol{\alpha}_1^{(k)},\boldsymbol{\alpha}_{|[\ell]|}^{(\ell)}} \\ \vdots & & \vdots \\ G_{\boldsymbol{\alpha}_{|[k]|}^{(k)},\boldsymbol{\alpha}_1^{(\ell)}} & \dots & G_{\boldsymbol{\alpha}_{|[k]|}^{(k)},\boldsymbol{\alpha}_{|[\ell]|}^{(\ell)}} \end{pmatrix}$$

$$G_{\boldsymbol{\alpha}_i^{(k)},\boldsymbol{\alpha}_j^{(\ell)}} := \int_\Omega \boldsymbol{x}^{\boldsymbol{\alpha}_i^{(k)}+\boldsymbol{\alpha}_j^{(\ell)}}\mathrm{d}\mu(\boldsymbol{x}) \in \mathbb{R}$$

Cholesky factorization

- If $\det G^{[\ell]} \neq 0$ for all $\ell = 0, 1, \dots$ then G admits the following Cholesky type factorization

$$G = S^{-1} H \left(S^{-1}\right)^{\top}$$

with

$$S^{-1} = \begin{pmatrix} \mathbb{I} & 0 & 0 & \cdots \\ (S^{-1})_{[1],[0]} & \mathbb{I} & 0 & \cdots \\ (S^{-1})_{[2],[0]} & (S^{-1})_{[2],[1]} & \mathbb{I} & \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

$$H = \begin{pmatrix} H_{[0]} & 0 & 0 & \cdots \\ 0 & H_{[1]} & 0 & \cdots \\ 0 & 0 & H_{[2]} & \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

where $H_{[k]}$ are symmetric matrices, $H_{[k]}^{\top} = H_{[k]}$

- When $\det G^{[\ell]} \neq 0$ the Cholesky type factorization holds and

$$\det G^{[\ell]} = \prod_{k=0}^{\ell-1} \det H_{[k]} \neq 0$$

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- The square matrices $H_{[k]} \in \mathbb{R}^{|[k]| \times |[k]|}$ can be expressed in terms of Schur complements of the truncated moment matrix as follows

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- If the quasi-determinants of the truncated moment matrices are invertible $\det \Theta_*(G^{[k+1]}) \neq 0$ the Cholesky factorization can be performed where

$$H_{[k]} = \Theta_*(G^{[k+1]})$$

Schur complements and quasi-determinants

Schur complements and quasi-determinants

The Schur complement (Emilie Haynsworth, 1968) ☺

Given $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in block form the Schur complement with respect to A (if $\det A \neq 0$) is

$$\operatorname{SC}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \equiv M/A := D - CA^{-1}B$$

Schur lemma in a disguise form $\det M = \det(A)\det(M/A)$ (If $[A, C] = 0$ then $\det M = \det(AD - BC)$)


Quasi-determinant???

Given any partitioned matrix

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,k} \\ A_{2,1} & A_{2,2} & \dots & A_{2,k} \\ \vdots & \vdots & & \vdots \\ A_{k,1} & A_{k,2} & \dots & A_{k,k} \end{pmatrix}$$

where $A_{i,j} \in \mathbb{R}^{m_i \times m_j}$ for $i, j \in \{1, \dots, k-1\}$, and $A_{k,k} \in \mathbb{R}^{\kappa_1 \times \kappa_2}$,$A_{i,k} \in \mathbb{R}^{m_i \times \kappa_2}$ and $A_{k,j} \in \mathbb{R}^{\kappa_1 \times m_j}$, we consider its quasi-determinant *à la Olver* (2006) recursively. In the Gel'fand style we require to all blocks to be regular squared equally sized matrices

Some history

In the late 1920 Archibald Richardson, one of the two responsible of Littlewood–Richardson rule, and the famous logician Arend Heyting, founder of intuitionist logic, studied possible extensions of the determinant notion to division rings. Heyting defined the *designant* of a matrix with noncommutative entries, which for 2×2 matrices was the Schur complement, and generalized to larger dimensions by induction.

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The situation today

A definitive impulse to the modern theory, 1990 till today, was given by Gel'fand, Rektah and collaborators. Quasi-determinants where defined over free division rings and was early noticed that **is not an analog of the commutative determinant but rather of a ratio determinants**. A cornerstone for quasi-determinants is the *heredity principle*, **quasi-determinants of quasi-determinants are quasi-determinants**; there is no analog of such a principle for determinants

The easiest quasi-determinant: a Schur complement

We start with $k = 2$, so that $A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$, in this case the first quasi-determinant $\Theta_1(A) := A/A_{1,1}$; i. e., a Schur complement which requires $\det A_{1,1} \neq 0$

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Olver vs Gel'fand

The notation of Olver is different to that of the Gel'fand school where $\Theta_1(A)=|A|_{2,2}=\begin{vmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & \boxed{A_{2,2}}\end{vmatrix}$. There is another quasi-determinant $\Theta_2(A)=A/A_{22}=|A|_{1,1}=\begin{vmatrix} \boxed{A_{1,1}} & A_{1,2} \\ A_{2,1} & A_{2,2}\end{vmatrix}$, the other Schur complement, and we need $A_{2,2}$ to be a invertible square matrix. Other quasi-determinants that can be considered for regular square blocks are $\begin{vmatrix} A_{1,1} & A_{1,2} \\ \boxed{A_{2,1}} & A_{2,2}\end{vmatrix}$ and $\begin{vmatrix} A_{1,1} & \boxed{A_{1,2}} \\ A_{2,1} & A_{2,2}\end{vmatrix}$. This last two quasi-determinants are out of the scope of the talk, as the partitioned matrix considered here have rectangular off diagonal blocks and therefore are not invertible

Example, consider

$$A = \left(\begin{array}{c|c:c} A_{1,1} & A_{1,2} & A_{1,3} \\ \hline A_{2,1} & A_{2,2} & A_{2,3} \\ \hdashline A_{3,1} & A_{3,2} & A_{3,3} \end{array}\right)$$

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$$\Theta_1(A)=\begin{vmatrix} A_{11,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & \boxed{\begin{matrix} A_{2,2} & A_{2,3} \\ A_{3,2} & A_{3,3} \end{matrix}} \\ A_{3,1} & & \end{vmatrix}=\begin{pmatrix} A_{2,2} & A_{2,3} \\ A_{3,2} & A_{3,3} \end{pmatrix}-\begin{pmatrix} A_{2,1} \\ A_{3,1} \end{pmatrix} A_{1,1}^{-1}\begin{pmatrix} A_{1,2} & A_{1,3} \end{pmatrix}$$

$$=\left(\begin{array}{c:c} A_{2,2}-A_{2,1}A_{1,1}^{-1}A_{1,2} & A_{2,3}-A_{2,1}A_{1,1}^{-1}A_{1,3} \\ \hdashline A_{3,2}-A_{3,1}A_{1,1}^{-1}A_{1,2} & A_{3,3}-A_{3,1}A_{1,1}^{-1}A_{1,3} \end{array}\right),$$

Take the quasi-determinant given by the Schur complement as indicated by the dashed lines

$$\Theta_2(\Theta_1(A)) = \begin{vmatrix} A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2} & A_{2,3} - A_{2,1}A_{1,1}^{-1}A_{1,3} \\ A_{3,2} - A_{3,1}A_{1,1}^{-1}A_{1,2} & \boxed{A_{3,3} - A_{3,1}A_{1,1}^{-1}A_{1,3}} \end{vmatrix}$$

$$= A_{3,3} - A_{3,1}A_{1,1}^{-1}A_{1,3}$$

$$- \big(A_{3,2} - A_{3,1}A_{1,1}^{-1}A_{1,2}\big)\big(A_{2,2} - A_{2,1}A_{1,1}^{-1}A_{1,2}\big)^{-1}\big(A_{2,3} - A_{2,1}A_{1,1}^{-1}A_{1,3}\big)$$

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Compute, for the very same matrix $A = \left(\begin{array}{cc|c} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ \hline A_{3,1} & A_{3,2} & A_{3,3} \end{array}\right)$ the Schur complement indicated by the non-dashed lines

$$\Theta_{\{1,2\}}(A) = \begin{vmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{1,3} & A_{2,3} & \boxed{A_{3,3}} \end{vmatrix}$$

$$= A_{3,3} - \begin{pmatrix} A_{3,1} & A_{3,2} \end{pmatrix} \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}^{-1} \begin{pmatrix} A_{1,3} \\ A_{2,3} \end{pmatrix}$$

From

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}^{-1} = \begin{pmatrix} A_{1,1}^{-1} + A_{1,1}^{-1} A_{1,2}(A_{2,2} - A_{2,1} A_{1,1}^{-1} A_{1,2})^{-1} A_{2,1} A_{1,1}^{-1} & -A_{1,1}^{-1} A_{1,2}(A_{2,2} - A_{2,1} A_{1,1}^{-1} A_{1,2}) \\ -(A_{2,2} - A_{2,1} A_{1,1}^{-1} A_{1,2})^{-1} A_{2,1} A_{1,1}^{-1} & (A_{2,2} - A_{2,1} A_{1,1}^{-1} A_{1,2})^{-1} \end{pmatrix}$$

$$\Rightarrow \Theta_2(\Theta_1(A)) = \Theta_{\{1,2\}}(A)$$

Heredity Principle

Quasi-determinants of quasi-determinants are quasi-determinants

Given any set $I = \{i_1, \dots, i_m\} \subset \{1, \dots, k\}$ the heredity principle allows us to define the quasi-determinant

$$\Theta_I(A) = \Theta_{i_1}(\Theta_{i_2}(\cdots \Theta_{i_m}(A) \cdots))$$

The ℓ-th quasi-determinant is

$$\Theta^{(\ell)}(A)=\Theta_{\{1,\dots,\ell-1,\ell+1,\dots,k\}}(A)=|A|_{\ell,\ell}=\begin{vmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,\ell} & \dots & A_{1,k}\\ A_{2,1} & A_{2,2} & \dots & A_{2,\ell} & \dots & A_{2,k}\\ \vdots & \vdots & & \vdots & & \vdots\\ A_{\ell,1} & A_{\ell,2} & \dots & \boxed{A_{\ell,\ell}} & \dots & A_{\ell,k}\\ \vdots & \vdots & & \vdots & & \vdots\\ A_{k,1} & A_{k,2} & \dots & A_{k,\ell} & \dots & A_{k,k}\end{vmatrix}$$

Last quasi-determinant ☺

$$\Theta_*(A)=\Theta^{(k)}(A)=|A|_{k,k}=\begin{vmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,k}\\ A_{2,1} & A_{2,2} & \dots & A_{2,k}\\ \vdots & \vdots & & \vdots\\ A_{k,1} & A_{k,2} & \dots & \boxed{A_{k,k}}\end{vmatrix}$$

Quasi-tau matrices

- In the 1D scenario the tau functions can be introduced as the determinant of a truncated moment matrix

$$\tau_k := \det G^{[k]}, \qquad k = 1, 2, \dots, \qquad \tau_0 = 1$$

And the relation with H_k is

$$H_k = \frac{\tau_{k+1}}{\tau_k}, \qquad \tau_{k+1} = H_k H_{k-1} \cdots H_0$$

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- $H_k = \det(G^{[k+1]}/G^{[k]}) = \Theta_*(G^{[k+1]})$
- The 1D scenario suggests that the squared norms H_k can be considered as **quasi-tau functions** (being the tau functions $\tau_k = \det G^{[k]}$ determinants of the truncated moment matrix and the quasi-tau functions $H_k = \Theta_*(G^{[k+1]})$ quasi-determinants of the truncated moment matrix)

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- This extends to the multivariant setting and now we have $H_{[k]} = \Theta_*(G^{[k+1]})$

☺

The polynomials

The MVOPR associated to the measure $\mathrm{d}\mu$ are

$$P = S\chi = \begin{pmatrix} P_{[0]} \\ P_{[1]} \\ \vdots \end{pmatrix}$$

$$P_{[k]}(\boldsymbol{x}) = \sum_{\ell=0}^{k} S_{[k],[\ell]}\chi_{[\ell]}(\boldsymbol{x}) = \begin{pmatrix} P_{\boldsymbol{\alpha}_1^{(k)}} \\ \vdots \\ P_{\boldsymbol{\alpha}_{|[k]|}^{(k)}} \end{pmatrix}$$

$$P_{\boldsymbol{\alpha}_i^{(k)}} = \sum_{\ell=0}^{k}\sum_{j=1}^{|[\ell]|} S_{\boldsymbol{\alpha}_i^{(k)},\boldsymbol{\alpha}_j^{(\ell)}}\boldsymbol{x}^{\boldsymbol{\alpha}_j^{(\ell)}}$$

The MVOPR satisfy for $\ell = 0, 1, \dots, k-1$ the following relations

$$\int_\Omega P_{[k]}(\boldsymbol{x})\mathrm{d}\mu(\boldsymbol{x})(P_{[\ell]}(\boldsymbol{x}))^\top = \int_\Omega P_{[k]}(\boldsymbol{x})\mathrm{d}\mu(\boldsymbol{x})(\chi_{[\ell]}(\boldsymbol{x}))^\top = 0$$

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Orthogonality conditions

For $\ell = 0, 1, \dots, k-1$, $i = 1, \dots, |[k]|$, $j = 1, \dots, |[\ell]|$

$$\int_\Omega P_{\boldsymbol{\alpha}_i^{(k)}}(\boldsymbol{x})P_{\boldsymbol{\alpha}_j^{(\ell)}}(\boldsymbol{x})\mathrm{d}\mu(\boldsymbol{x}) = \int_\Omega P_{\boldsymbol{\alpha}_i^{(k)}}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{\alpha}_j^{(\ell)}}\mathrm{d}\mu(\boldsymbol{x}) = 0$$

with the normalization conditions ($i, j = 1, \dots, |[k]|$)

$$\int_\Omega P_{\boldsymbol{\alpha}_i}(\boldsymbol{x})P_{\boldsymbol{\alpha}_j}(\boldsymbol{x})\mathrm{d}\mu(\boldsymbol{x}) = \int_\Omega P_{\boldsymbol{\alpha}_i}(\boldsymbol{x})\boldsymbol{x}^{\boldsymbol{\alpha}_j}\mathrm{d}\mu(\boldsymbol{x}) = H_{\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_j}$$

Quasi-determinantal expressions

Schur complements

The MVOPR can be expressed as Schur complements of bordered truncated moment matrices

$$P_{[\ell]}(\boldsymbol{x}) = \mathrm{SC}\left(\begin{array}{ccc|c} G_{[0],[0]} & \cdots & G_{[0],[\ell-1]} & \chi_{[0]}(\boldsymbol{x}) \\ \vdots & & \vdots & \vdots \\ G_{[\ell-1],[0]} & \cdots & G_{[\ell-1],[\ell-1]} & \chi_{[\ell-1]}(\boldsymbol{x}) \\ \hline G_{[\ell],[0]} & \cdots & G_{[\ell],[\ell-1]} & \chi_{[\ell]}(\boldsymbol{x}) \end{array}\right)$$

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Quasi-determinants ☺

$$P_{[\ell]}(\boldsymbol{x}) = \Theta_{*}\begin{pmatrix} G_{[0],[0]} & \cdots & G_{[0],[\ell-1]} & \chi_{[0]}(\boldsymbol{x}) \\ \vdots & & \vdots & \vdots \\ G_{[\ell],[0]} & \cdots & G_{[\ell],[\ell-1]} & \chi_{[\ell]}(\boldsymbol{x}) \end{pmatrix}$$

The shift matrices

Shift matrices are

$$\Lambda_a = \begin{pmatrix} 0 & (\Lambda_a)_{[0],[1]} & 0 & 0 & \cdots \\ 0 & 0 & (\Lambda_a)_{[1],[2]} & 0 & \cdots \\ 0 & 0 & 0 & (\Lambda_a)_{[2],[3]} & \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Given any $\boldsymbol{k} = \sum_{a=1}^{D} k_a \boldsymbol{e}_a \in \mathbb{Z}_+^D$ we define

$$\Lambda_{\boldsymbol{k}} := \Lambda_1^{k_1} \cdots \Lambda_D^{k_D}$$

For $D = 2$ we have

$$\Lambda_1 = \left(\begin{array}{c|cc|ccc|cccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \end{array}\right)$$

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0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots\\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & &
\end{array}\right)$$

$$\Lambda_2=\left(\begin{array}{c|cc|ccc|cccc|c}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots\\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & &
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$$
\Lambda_1\Lambda_2=\Lambda_2\Lambda_1=\left(\begin{array}{c|cc|ccc|cccc|c}
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\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
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\hline
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\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & &
\end{array}\right)
$$

Properties of the shift matrices

$$
\Lambda_k\Lambda_\ell=\Lambda_{k+\ell}=\Lambda_\ell\Lambda_k, \qquad \Lambda_k(\Lambda_k)^\top=\mathbb{I}
$$

$$
\boxed{\Lambda_k\chi(x)=x^k\chi(x)}
$$

String equation and Jacobi matrices

The multivariate string equation ☺

The moment matrix G satisfies $\forall \boldsymbol{k} \in \mathbb{Z}_+^D$

$$\boxed{\Lambda_{\boldsymbol{k}} G = G (\Lambda_{\boldsymbol{k}})^\top}$$

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Properties of Jacobi matrices

The Jacobi type matrices satisfy $\forall \boldsymbol{k} \in \mathbb{Z}_+^D$

$$J_{\boldsymbol{k}}^\top = H^{-1} J_{\boldsymbol{k}} H, \qquad \boxed{J_{\boldsymbol{k}} P = \boldsymbol{x}^{\boldsymbol{k}} P}$$

The explicit form of the basic Jacobi matrices is

$$J_a = \begin{pmatrix} (J_a)_{[0],[0]} & (J_a)_{[0],[1]} & 0 & 0 & 0 & \cdots \\ (J_a)_{[1],[0]} & (J_a)_{[1],[1]} & (J_a)_{[1],[2]} & 0 & 0 & \cdots \\ 0 & (J_a)_{[2],[1]} & (J_a)_{[2],[2]} & (J_a)_{[2],[3]} & 0 & \cdots \\ 0 & 0 & (J_a)_{[3],[2]} & (J_a)_{[3],[3]} & (J_a)_{[3],[4]} & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix}$$

where

$$\begin{aligned} (J_a)_{[k],[k-1]} &= H_{[k]} \left[(\Lambda_a)_{[k-1],[k]} \right]^\top \left(H_{[k-1]} \right)^{-1}, \\ (J_a)_{[k],[k]} &= \beta_{[k]} (\Lambda_a)_{[k-1],[k]} - (\Lambda_a)_{[k],[k+1]} \beta_{[k+1]}, \\ (J_a)_{[k],[k+1]} &= (\Lambda_a)_{[k],[k+1]} \end{aligned}$$

For $D=2$ we have

$$J_1=\left(\begin{array}{c|cc|ccc|cccc|c}
\spadesuit & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
\hline
\clubsuit & \spadesuit & \spadesuit & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
\clubsuit & \spadesuit & \spadesuit & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots\\
\hline
0 & \clubsuit & \clubsuit & \spadesuit & \spadesuit & \spadesuit & 1 & 0 & 0 & 0 & \dots\\
0 & \clubsuit & \clubsuit & \spadesuit & \spadesuit & \spadesuit & 0 & 1 & 0 & 0 & \dots\\
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0 & \clubsuit & \clubsuit & \spadesuit & \spadesuit & \spadesuit & 0 & 1 & 0 & 0 & \dots\\
0 & \clubsuit & \clubsuit & \spadesuit & \spadesuit & \spadesuit & 0 & 0 & 1 & 0 & \dots\\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & &
\end{array}\right)$$

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\spadesuit & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots\\
\hline
\clubsuit & \spadesuit & \spadesuit & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots\\
\clubsuit & \spadesuit & \spadesuit & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots\\
\hline
0 & \clubsuit & \clubsuit & \spadesuit & \spadesuit & \spadesuit & 0 & 1 & 0 & 0 & \dots\\
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0 & \clubsuit & \clubsuit & \spadesuit & \spadesuit & \spadesuit & 0 & 0 & 0 & 1 & \dots\\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & &
\end{array}\right)$$

We introduce the following objects

$$\boldsymbol{\Lambda} := (\Lambda_1, \dots, \Lambda_D)^\top, \qquad \boldsymbol{J} := (J_1, \dots, J_D)^\top$$

for $\boldsymbol{n} = (n_1, \dots, n_D)^\top \in \mathbb{R}^D$ we define the following *dot products*

$$\boldsymbol{n} \cdot \boldsymbol{\Lambda} := \sum_{a=1}^{D} n_a \Lambda_a, \qquad \boldsymbol{n} \cdot \boldsymbol{J} := \sum_{a=1}^{D} n_a J_a.$$

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Three term relation ☺

The celebrated Xu's three term relations for $k = 1, 2, \dots$ are

$$\begin{aligned}(\boldsymbol{n}\cdot\boldsymbol{x})P_{[k]}(\boldsymbol{x}) = H_{[k]}(\boldsymbol{n}\cdot\boldsymbol{\Lambda})^\top_{[k-1],[k]}H^{-1}_{[k-1]}P_{[k-1]}(\boldsymbol{x}) \\ + \big(\beta_{[k]}(\boldsymbol{n}\cdot\boldsymbol{\Lambda})_{[k-1],[k]} - (\Lambda_a)_{[k],[k+1]}\beta_{[k+1]}\big)P_{[k]}(\boldsymbol{x}) \\ + (\boldsymbol{n}\cdot\boldsymbol{\Lambda})_{[k],[k+1]}P_{[k+1]}(\boldsymbol{x})\end{aligned}$$

In the $2D$ case with $\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$

$$\boldsymbol{n}\cdot\boldsymbol{\Lambda} = \left(\begin{array}{c|cc|ccc|cccc|c}
0 & n_1 & n_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\hline
0 & 0 & 0 & n_1 & n_2 & 0 & 0 & 0 & 0 & 0 & \dots \\
0 & 0 & 0 & 0 & n_1 & n_2 & 0 & 0 & 0 & 0 & \dots \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_1 & n_2 & \dots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & &
\end{array}\right)$$

Christoffel–Darboux kernels

$$K^{(\ell)}(\boldsymbol{x},\boldsymbol{y}) := \left(\chi^{[\ell]}(\boldsymbol{x})\right)^\top \left(G^{[\ell]}\right)^{-1}\chi^{[\ell]}(\boldsymbol{y}) = \sum_{k=0}^{\ell-1}\left(P_{[k]}(\boldsymbol{x})\right)^\top \left(H_{[k]}\right)^{-1} P_{[k]}(\boldsymbol{y})$$

Christoffel–Darboux formula ☺

$$\begin{aligned}(\boldsymbol{n}\cdot(\boldsymbol{x}-\boldsymbol{y}))K^{(\ell)}(\boldsymbol{x},\boldsymbol{y}) = & \left((\boldsymbol{n}\cdot\boldsymbol{\Lambda})_{[\ell-1],[\ell]}P_{[\ell]}(\boldsymbol{x})\right)^\top \left(H_{[\ell-1]}\right)^{-1}P_{[\ell-1]}(\boldsymbol{y}) \\ & - P_{[\ell-1]}(\boldsymbol{x})^\top \left(H_{[\ell-1]}\right)^{-1}(\boldsymbol{n}\cdot\boldsymbol{\Lambda})_{[\ell-1],[\ell]}P_{[\ell]}(\boldsymbol{y})\end{aligned}$$

The discrete Toda flows

- D discrete flows we consider an invertible matrix

$$N = (n_{a,b})_{a,b=1,\dots,D} \in \mathrm{GL}(\mathbb{R}^D),$$

and therefore D linearly independent vectors $\boldsymbol{n}_a = (n_{a,1}, \dots, n_{a,D})^\top$, and a vector $\boldsymbol{q} = (q_1, \dots, q_D)^\top \in \mathbb{R}^D$, where $q_a \neq 0$, $a = \{1, \dots, D\}$

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- For each multi-index $\boldsymbol{m}=(m_1,\dots,m_D)^\top\in\mathbb{Z}^D$ we consider the measure

$$\mathrm{d}\mu_{\boldsymbol{m}}(\boldsymbol{x})=\Big[\prod_{a=1}^{D}(\boldsymbol{n}_a\cdot\boldsymbol{x}-q_a)^{m_a}\Big]\mathrm{d}\mu(\boldsymbol{x}).$$

- We introduce the bounded open convex polytope

$$R := \{\boldsymbol{x} \in \mathbb{R}^D : |\boldsymbol{n}_1 \cdot \boldsymbol{x}| < |q_1|, \dots, |\boldsymbol{n}_D \cdot \boldsymbol{x}| < |q_D|\}$$

and sets $R_a := \{\boldsymbol{x} \in \mathbb{R}^D : -|q_a| < \boldsymbol{n}_a \cdot \boldsymbol{x} < |q_a|\}$, $a \in \{1, \dots, D\}$

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- $\mathrm{d}\mu_{\boldsymbol{m}}$ has a definite sign in $R \cap \operatorname{supp}(\mu)$
- If $\operatorname{supp} \mu \subset R$, then the moment matrices $G(\boldsymbol{m})$ of the measures $\mathrm{d}\mu_{\boldsymbol{m}}$ satisfy

$$\boxed{G(\boldsymbol{m}) = \Big(\prod_{a=1}^{D} (\boldsymbol{n}_a \cdot \boldsymbol{\Lambda} - q_a)^{m_a}\Big) G = G\Big(\prod_{a=1}^{D} (\boldsymbol{n}_a \cdot \boldsymbol{\Lambda}^\top - q_a)^{m_a}\Big)}$$

- Undressed wave matrices

$$W_0(\boldsymbol{m}) := \prod_{a=1}^{D} (\boldsymbol{n}_a \cdot \boldsymbol{\Lambda} - q_a)^{m_a}$$

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- Translations and partial difference

$$(T_a f)(m_1, \dots, m_a, \dots, m_D) = f(m_1, \dots, m_a + 1, \dots, m_D)$$
$$\Delta_a := T_a - 1$$

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$$\Delta_a := T_a - 1$$

- We introduce the following adjoint lattice resolvents

$$\begin{aligned} M_a &:= S(T_a S)^{-1} \\ &= H\big((T_a S)(\boldsymbol{n}_a \cdot \boldsymbol{\Lambda} - q_a) S^{-1}\big)^{\top} (T_a H)^{-1} \\ &= \mathbb{I} + \rho_a \end{aligned}$$

with $\rho_a = H(\boldsymbol{n}_a \cdot \boldsymbol{\Lambda})^{\top} (T_a H)^{-1}$

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- Lattice resolvents for each $a \in \{1, \dots, D\}$:

$$\begin{aligned}\omega_a &:= (T_a H) M_a^\top H^{-1} \\ &= (T_a S)(\boldsymbol{n}_a \cdot \boldsymbol{\Lambda} - q_a) S^{-1} \\ &= \alpha_a + \boldsymbol{n}_a \cdot \boldsymbol{\Lambda}\end{aligned}$$

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The quasi-tau matrices $H_{[k]}$ are subject to the following

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Discrete Toda equations!

$$\Delta_b\big((\Delta_a H_{[k]})H_{[k]}^{-1}\big) = (\boldsymbol{n}_a\cdot\boldsymbol{\Lambda})_{[k],[k+1]}H_{[k+1]}[(\boldsymbol{n}_b\cdot\boldsymbol{\Lambda})_{[k],[k+1]}]^\top\big(T_b H_{[k]}\big)^{-1} \\ -(T_a H_{[k]})[(\boldsymbol{n}_b\cdot\boldsymbol{\Lambda})_{[k-1],[k]}]^\top\big(T_a T_b H_{[k-1]}\big)^{-1}(\boldsymbol{n}_a\cdot\boldsymbol{\Lambda})_{[k-1],[k]}$$

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Discrete Toda equations!

$$\overbrace{\Delta_b\big((\Delta_a H_{[k]})H_{[k]}^{-1}\big)}^{\text{second partial difference}} = \overbrace{(\boldsymbol{n}_a\cdot\boldsymbol{\Lambda})_{[k],[k+1]}H_{[k+1]}[(\boldsymbol{n}_b\cdot\boldsymbol{\Lambda})_{[k],[k+1]}]^\top\big(T_bH_{[k]}\big)^{-1}}^{\text{quadratic term in } H}$$
$$-\underbrace{(T_aH_{[k]})[(\boldsymbol{n}_b\cdot\boldsymbol{\Lambda})_{[k-1],[k]}]^\top\big(T_aT_bH_{[k-1]}\big)^{-1}(\boldsymbol{n}_a\cdot\boldsymbol{\Lambda})_{[k-1],[k]}}_{\text{quadratic term in } H}$$

- If T_aG and G admit Cholesky $\Rightarrow$ LU factorization

$$\boxed{\boldsymbol{n}_a \cdot \boldsymbol{J} - q_a = M_a \omega_a}$$

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- In terms of quasi-determinants of the Jacobi matrices

$$\rho_{a,[k]}=(\boldsymbol{n}_a\cdot\boldsymbol{J}_{[k],[k-1]})\big(\Theta_*(\boldsymbol{n}_a\cdot\boldsymbol{J}^{[k]}-q_a\mathbb{I}^{[k]})\big)^{-1}$$
$$\alpha_{a,[k]}=\Theta_*(\boldsymbol{n}_a\cdot\boldsymbol{J}^{[k+1]}-q_a\mathbb{I}^{[k+1]})$$

Integrability

1. The wave matrices W_1 and W_2 both satisfy the following linear system

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3. Zakharov–Shabat equations

$$(T_a\omega_b)\omega_a = (T_b\omega_a)\omega_b, \qquad M_a(T_a M_b) = M_b(T_b M_a)$$

Quasi-tau functions formulæ for MOVPR

- From $M_a T_a P = P$ and $\omega_a P = (\boldsymbol{n}\cdot\boldsymbol{x} - q_a) T_a P$ we get

$$\rho_{a,[k]}(T_a P)_{[k-1]} + (T_a P)_{[k]} = P_{[k]},$$
$$\alpha_{a,[k-1]} P_{[k-1]} + (\boldsymbol{n}_a \cdot \boldsymbol{\Lambda})_{[k-1],[k]} P_{[k]} = (\boldsymbol{n}_a \cdot \boldsymbol{x} - q_a)(T_a P)_{[k-1]}$$

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- When $\boldsymbol{p}\in\pi_a^+$, i.e. $\boldsymbol{n}_a\cdot\boldsymbol{p} = q_a$, the **important** relation appears

$$\boxed{(\boldsymbol{n}_a\cdot\boldsymbol{\Lambda})_{[k-1],[k]} P_{[k]}(\boldsymbol{p}) = -\alpha_{a,[k-1]} P_{[k-1]}(\boldsymbol{p})}$$

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$$\begin{aligned}\rho_{a,[k]}(T_a P)_{[k-1]} + (T_a P)_{[k]} &= P_{[k]},\\ \alpha_{a,[k-1]} P_{[k-1]} + (\boldsymbol{n}_a\cdot\boldsymbol{\Lambda})_{[k-1],[k]} P_{[k]} &= (\boldsymbol{n}_a\cdot\boldsymbol{x} - q_a)(T_a P)_{[k-1]}\end{aligned}$$

- When $\boldsymbol{p}\in\pi_a^+$, i.e. $\boldsymbol{n}_a\cdot\boldsymbol{p} = q_a$, the **important** relation appears

$$(\boldsymbol{n}_a\cdot\boldsymbol{\Lambda})_{[k-1],[k]} P_{[k]}(\boldsymbol{p}) = -\alpha_{a,[k-1]} P_{[k-1]}(\boldsymbol{p})$$

- In the 1D scenario we have for $p = q/n$ that $P_k(p) = -\alpha_{k-1} P_{k-1}(p)$, and iterating this relation one gets

$$P_k(q) = (-1)^k \frac{TH_{k-1} TH_{k-2}\cdots TH_0}{H_{k-1} H_{k-2}\cdots H_0} = (-1)^k \frac{T\tau_k}{\tau_k}$$

This is a well known expression in terms of Miwa shifts of τ-functions

$$(\boldsymbol{n}_a \cdot \boldsymbol{\Lambda})_{[k-1],[k]} P_{[k]}(\boldsymbol{p}) = -\alpha_{a,[k-1]} P_{[k-1]}(\boldsymbol{p})$$

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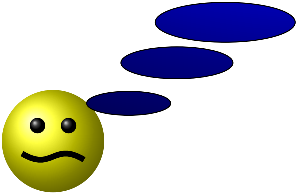
Is it possible to clear $P_{[k]}(\boldsymbol{x})$ having $(\boldsymbol{n}_a \cdot \boldsymbol{\Lambda})_{[k-1],[k]}$ not left inverse?



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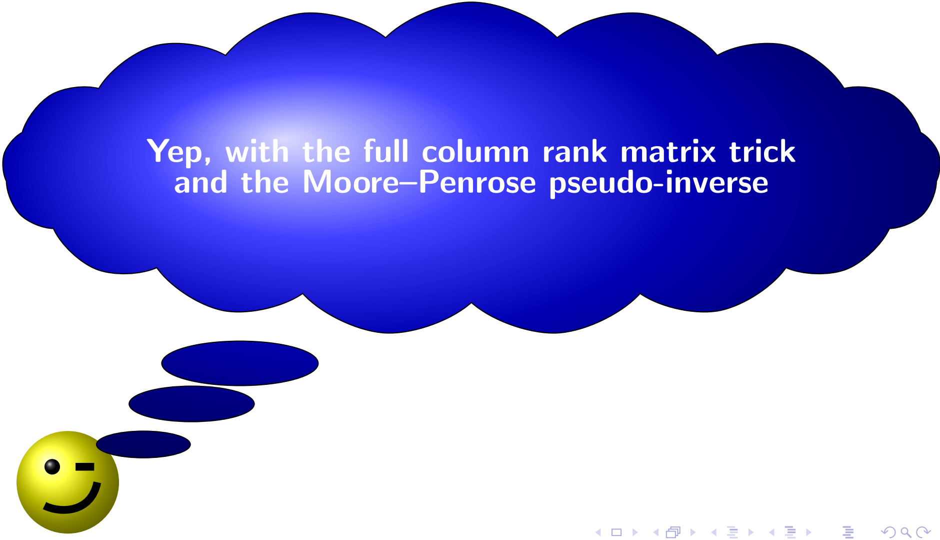

- The matrices $(\boldsymbol{n}_a \cdot \boldsymbol{\Lambda})_{[k],[k+1]}$ have a right inverse but do not have a left inverse, but if we arrange all of them together we get

$$[N\boldsymbol{\Lambda}]_k := \begin{pmatrix} (\boldsymbol{n}_1 \cdot \boldsymbol{\Lambda})_{[k],[k+1]} \\ \vdots \\ (\boldsymbol{n}_D \cdot \boldsymbol{\Lambda})_{[k],[k+1]} \end{pmatrix} \in \mathbb{R}^{D|[k]|\times|[k+1]|}$$

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- $[N\boldsymbol{\Lambda}]_k$ is a full column rank matrix $\Rightarrow$ the correlation matrix $[N\boldsymbol{\Lambda}]_k^\top [N\boldsymbol{\Lambda}]_k \in \mathbb{R}^{|[k+1]|\times|[k+1]|}$ is invertible and the Moore–Penrose pseudo-inverse is

$$[N\boldsymbol{\Lambda}]_k^+ = \big([N\boldsymbol{\Lambda}]_k^\top [N\boldsymbol{\Lambda}]_k\big)^{-1} [N\boldsymbol{\Lambda}]_k^\top$$

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- The clearing leads to

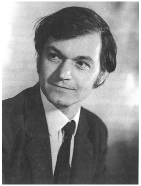
$$P_{[k]}(\boldsymbol{q}) = -[N\boldsymbol{\Lambda}]_{(k-1)}^+ [\boldsymbol{T}^{(N\boldsymbol{q})}H]_{k-1} H_{[k-1]}^{-1} P_{[k-1]}(\boldsymbol{q})$$

where $\boldsymbol{q} := (q_1, \dots, q_d)$, $[\boldsymbol{T}^{(\boldsymbol{q})}H]_k := \begin{pmatrix} T_1^{(\boldsymbol{q})}H_{[k]} \\ \vdots \\ T_D^{(\boldsymbol{q})}H_{[k]} \end{pmatrix} \in \mathbb{R}^{D|[k]|\times|[k]|}$

Quasi-tau and MOVPR ☺

The MVOPR can be expressed in terms of quasi-tau matrices H and its discrete time translations as follows

$$P_{[k]}(\boldsymbol{q}) = (-1)^k [N\boldsymbol{\Lambda}]^{+}_{k-1} [\boldsymbol{T}^{(N\boldsymbol{q})} H]_{k-1} (H_{[k-1]})^{-1} \times [N\boldsymbol{\Lambda}]^{+}_{k-2} [\boldsymbol{T}^{(N\boldsymbol{q})} H]_{k-2} (H_{[k-2]})^{-1} \cdots [N\boldsymbol{\Lambda}]^{+}_{0} [\boldsymbol{T}^{(N\boldsymbol{q})} H]_{0} H_{[0]}^{-1}$$



Bravo!
Another application of our generalised inverse

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This result justifies once again the quasi-tau denomination for $H_{[k]}$ as it reproduces the 1D result. We have not found such an extension for the τ function

Multivariate elementary Darboux transformations

OPRL's elementary Darboux transformations

For $D = 1$ the important relation derived previously leads to Darboux transformations

$$P_k(q) = -\alpha_k P_{k-1}(q) \quad \Longrightarrow \quad \boxed{\alpha_k = -\frac{P_k(q)}{P_{k-1}(q)}}$$

This is not true in $D > 1$

gives the so called kernel polynomials

$$TP_{k-1}(x) = \frac{1}{x-q}\Big(P_k(x) - P_k(q)\frac{1}{P_{k-1}(q)}P_{k-1}(x)\Big)$$

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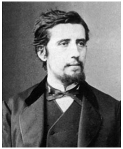

Lire mon article
Sur une proposition relative aux équations linéaires
publiés en 1882 dans Comptes Rendus

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It can be written as a quasi-determinant (a very trivial one indeed!!)

$$TP_{k-1}(x) = \frac{1}{x-q}\Theta_*\begin{pmatrix} P_{k-1}(q) & P_{k-1}(x) \\ P_k(q) & P_k(x) \end{pmatrix}$$

For $D > 1$ and $\boldsymbol{p}$ such that $\boldsymbol{n} \cdot \boldsymbol{p} = q$ we have

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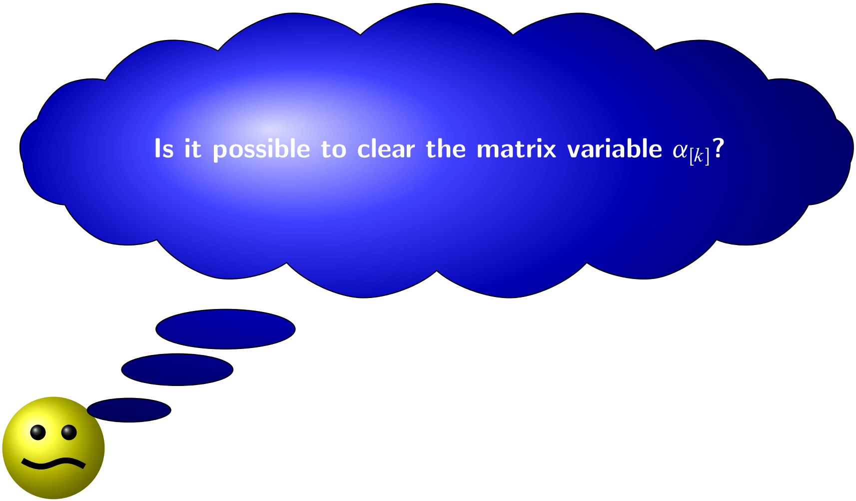

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Sample matrix trick

Nodes, sample matrices and poised sets

Given the set $\{\boldsymbol{p}_1,\dots \boldsymbol{p}_{|[k]|}\} \subset \pi^+ = \{\boldsymbol{x} \in \mathbb{R}^D : \boldsymbol{x}\cdot\boldsymbol{n} = q\} \subset \mathbb{R}^D$, whose elements are known as *nodes*, we consider the *sample matrices*

$$\Sigma^k_{[\ell]} = \begin{pmatrix} P_{[\ell]}(\boldsymbol{p}_1) & \dots & P_{[\ell]}(\boldsymbol{p}_{|[k]|}) \end{pmatrix} \in \mathbb{R}^{|[\ell]|\times|[k]|}$$

The set $\{\boldsymbol{p}_1,\dots \boldsymbol{p}_{|[k]|}\}$ of nodes is said to be a *poised set* for the *interpolation polynomials* $\{P_{\boldsymbol{k}_a}\}_{a=1}^{|[k]|}$ if the sample matrix $\Sigma^k_{[k]}$ is invertible, i.e. $\det \Sigma^k_{[k]} \neq 0$.

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For a poised set $\{\boldsymbol{p}_1,\dots,\boldsymbol{p}_{|[k]|}\}\subset\pi^+\subset\mathbb{R}^D$ of nodes we can write

$$\alpha_{[k]}=-(\boldsymbol{n}\cdot\boldsymbol{\Lambda})_{[k],[k+1]}\Sigma^k_{[k+1]}\big(\Sigma^k_{[k]}\big)^{-1}$$

The multivariate degree one Darboux transformation ☺

Given a poised set $\{\boldsymbol{p}_1,\dots \boldsymbol{p}_{|[k]|}\}\subset\pi^+\subset\mathbb{R}^D$ of nodes we have the following expressions of the degree one Darboux transformed MVOPR, the kernel polynomials $TP(\boldsymbol{x})$ associated with $(\boldsymbol{n}\cdot\boldsymbol{x}-q)\mathrm{d}\mu(\boldsymbol{x})$, in terms of quasi-determinants of the original MVOPR

$$(TP)_{[k]}(\boldsymbol{x})=(\boldsymbol{n}\cdot\boldsymbol{x}-q)^{-1}(\boldsymbol{n}\cdot\boldsymbol{\Lambda})_{[k],[k+1]}\Theta_*\begin{pmatrix}\Sigma^k_{[k]} & P_{[k]}(\boldsymbol{x})\\ \Sigma^k_{[k+1]} & P_{[k+1]}(\boldsymbol{x})\end{pmatrix}$$

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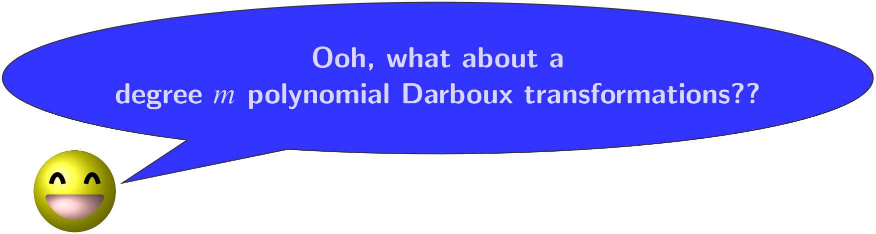
Quasi-tau matrices transform according

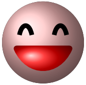
$$(TH)_{[k]}=(\boldsymbol{n}\cdot\boldsymbol{\Lambda})_{[k],[k+1]}\Theta_*\begin{pmatrix}\Sigma^k_{[k]} & H_{[k]}\\ \Sigma^k_{[k+1]} & 0\end{pmatrix}$$

Polynomial Darboux transformations



Polynomial Darboux transformations

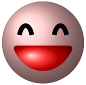
You mean $\mathrm{d}\mu(\boldsymbol{x}) \to \mathbb{Q}(\boldsymbol{x})\mathrm{d}\mu(\boldsymbol{x})$ **with**
with $\mathbb{Q}(\boldsymbol{x})$ a multivariate polynomial of $\deg\mathbb{Q} = m$??



Polynomial Darboux transformations

Polynomial Darboux transformations

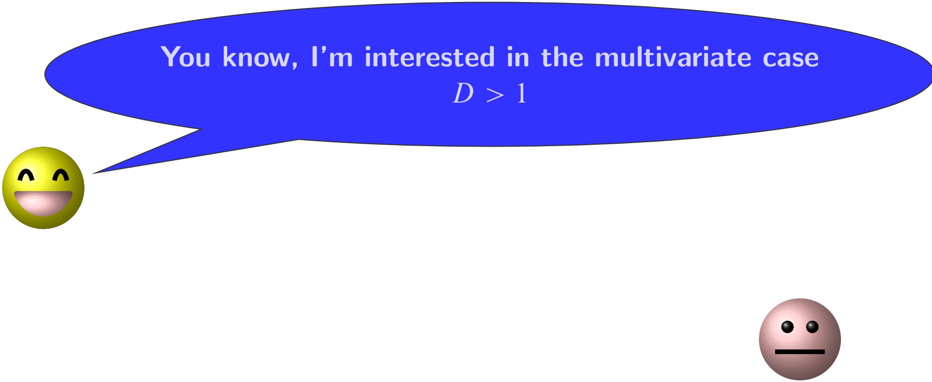
Well, as is pointed out in Gabor Szegő's book for $D = 1$ this was already answered for $\mathrm{d}\mu = \mathrm{d}x$ by Elwin Bruno Christoffel back in 1858



Polynomial Darboux transformations

Die tranſformierten Polynome ſind

$$\frac{1}{c(x-q_1)\cdots(x-q_m)}\begin{vmatrix} p_n(x) & \dots & p_{n+m}(x) \\ p_n(q_1) & \dots & p_{n+m}(q_1) \\ \vdots & & \vdots \\ p_n(q_m) & \dots & p_{n+m}(q_m) \end{vmatrix}$$


You know, I'm interested in the multivariate case
$D > 1$

Resolvents

- Degree m multivariate polynomial

$$\mathbb{Q} = \sum_{j=0}^{m} \mathbb{Q}^{(j)} \qquad \deg Q^{(j)} = j \qquad Q^{(m)} \neq 0$$

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- Polynomial Darboux transformation and resolvent

$$T\,\mathrm{d}\mu(\boldsymbol{x}) = \mathbb{Q}(\boldsymbol{x})\mathrm{d}\mu(x) \qquad \omega = (TS)\mathbb{Q}(\boldsymbol{\Lambda})S^{-1}$$

Structure of the resolvent

The degree m resolvent ω can be expressed in diagonals as follows

$$\omega = \underbrace{\mathbb{Q}^{(m)}(\boldsymbol{\Lambda})}_{m\text{-th superdiagonal}} + \underbrace{(T\beta)\mathbb{Q}^{(m-1)}(\boldsymbol{\Lambda}) - \mathbb{Q}^{(m-1)}(\boldsymbol{\Lambda})\beta}_{(m-1)\text{-th superdiagonal}} \\ \vdots \\ + \underbrace{(TH)H^{-1}}_{\text{diagonal}}$$

The sample matrix trick again ☺

- The MOVPR satisfy $\mathbb{Q}(\boldsymbol{x})TP(\boldsymbol{x}) = \omega P(\boldsymbol{x})$

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- For any element $\boldsymbol{p}$ in the zero set $Z_{\mathbb{Q}} := \{\boldsymbol{x} \in \mathbb{R}^D : \mathbb{Q}(\boldsymbol{x}) = 0\}$ we have the important relation

$$\omega_{[k],[k+m]} P_{[k+m]}(\boldsymbol{p}) + \omega_{[k],[k+1]} P_{[k+m-1]}(\boldsymbol{p}) + \cdots + \omega_{[k],[k]} P_{[k]}(\boldsymbol{p}) = 0$$

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- For the sets $\{\boldsymbol{p}_j^{(i)}\}_{j=1}^{|[k+i-1]|}$, $i \in \{1,\dots,m\}$, we use the notation $\Sigma_{[\ell]}^{(i),k}$ and we suppose that $\cup_{i=1}^m \{\boldsymbol{p}_j^{(i)}\}_{j=1}^{|[k-1+i]|} \subset \cup_{i=1}^m \pi^{(i),+}$ is a poised set for $\begin{pmatrix} P_{[k]}(\boldsymbol{x}) \\ \vdots \\ P_{[k+m-1]}(\boldsymbol{x}) \end{pmatrix}$, i.e. $\begin{vmatrix} \Sigma_{[k]}^{(1),k} & \dots & \Sigma_{[k]}^{(m),k+m-1} \\ \vdots & & \vdots \\ \Sigma_{[k+m-1]}^{(1),k} & \dots & \Sigma_{[k+m-1]}^{(m),k+m-1} \end{vmatrix} \neq 0$

Clearing the omegas

Then

$$
\begin{aligned}
&\begin{pmatrix}\omega_{[k],[k]} & \dots & \omega_{[k],[k+m-1]}\end{pmatrix}\\
&= -\omega_{[k],[k+m]}\begin{pmatrix}\Sigma^{(1),k}_{[k+m]} & \dots & \Sigma^{(m),k+m-1}_{[k+m]}\end{pmatrix}\begin{pmatrix}\Sigma^{(1),k}_{[k]} & \dots & \Sigma^{(m),k+m-1}_{[k]}\\ \vdots & & \vdots\\ \Sigma^{(1),k}_{[k+m-1]} & \dots & \Sigma^{(m),k+m-1}_{[k+m-1]}\end{pmatrix}^{-1}
\end{aligned}
$$

Multivariate quasi-determinantal Christoffel formula ☺

$$
TP_{[k]}(\boldsymbol{x}) = \frac{(\mathbb{Q}(\boldsymbol{\Lambda}))_{[k],[k+m]}}{\mathbb{Q}(\boldsymbol{x})}\Theta_*\begin{pmatrix} \Sigma^{(1),k}_{[k]} & \dots & \Sigma^{(m),k+m-1}_{[k]} & P_{[k]}(\boldsymbol{x}) \\ \vdots & & \vdots & \vdots \\ \Sigma^{(1),k}_{[k+m]} & \dots & \Sigma^{(m),k+m-1}_{[k+m]} & P_{[k+m]}(\boldsymbol{x}) \end{pmatrix}
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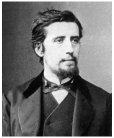

Irreducible polynomials and Darboux transformations

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- **The situation in $D > 1$ is radically different**. There are many irreducible polynomials of degree higher than one. Hence, an irreducible polynomial Darboux transformation is not reachable by iterations.
- Therefore, **degree one Darboux transformations do not deserve the name of elementary for $D > 1$.**

Continuous Toda flows

- Covector of time variables

$$t = (t_{[0]}, t_{[1]}, \dots), \qquad t_{[k]} = (t_{\boldsymbol{\alpha}_1^{(k)}}, \dots, t_{\boldsymbol{\alpha}_{|[k]|}^{(k)}}), \qquad t_{\boldsymbol{\alpha}_j^{(k)}} \in \mathbb{R}$$

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- Deformation matrix is

$$W_0(t, \boldsymbol{m}) = \exp\Big(\sum_{k=0}^{\infty}\sum_{j=1}^{|[k]|} t_{\boldsymbol{\alpha}_j^{(k)}} \Lambda_{\boldsymbol{\alpha}_j^{(k)}}\Big) \prod_{a=1}^{D} (\boldsymbol{n}_a \cdot \boldsymbol{\Lambda} - q_a)^{m_a}$$

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- Deformed moment matrix $G(t, \boldsymbol{m}) := W_0(t, \boldsymbol{m}) G$

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- The deformed moment matrix is the moment matrix of the following deformed measure

$$\mathrm{d}\mu_{t,\boldsymbol{m}}(\boldsymbol{x}) = \mathrm{e}^{t(\boldsymbol{x})}\,\mathrm{d}\mu_{\boldsymbol{m}}(\boldsymbol{x}) = \mathrm{e}^{t(\boldsymbol{x})}\Big[\prod_{a=1}^{D} (\boldsymbol{n}_a \cdot \boldsymbol{x} - q_a)^{m_a}\Big]\mathrm{d}\mu(\boldsymbol{x})$$

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$$W_1(t,\boldsymbol{m}) := S(t,\boldsymbol{m})W_0(t,\boldsymbol{m}), \quad W_2(t,\boldsymbol{m}) := H(t,\boldsymbol{m})\big((S(t,\boldsymbol{m}))^{-1}\big)^{\top}$$

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- The wave semi-infinite matrices are

$$W_1(t,\boldsymbol{m}) := S(t,\boldsymbol{m}) W_0(t,\boldsymbol{m}), \quad W_2(t,\boldsymbol{m}) := H(t,\boldsymbol{m})\big((S(t,\boldsymbol{m}))^{-1}\big)^\top$$

- Baker functions are defined by

$$\Psi_1 := W_1 \chi, \qquad \Psi_2 := W_2 \chi^*,$$

  while adjoint Baker functions are given by

$$\Psi_1^* := (W_1^{-1})^\top \chi^*, \qquad \Psi_2^* := (W_2^{-1})^\top \chi.$$

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- Baker functions in terms of MVOPR and its multivariate Cauchy transforms (second kind functions)

$$(\Psi_1)_{\boldsymbol{\alpha}_i}(\boldsymbol{z}) = \mathrm{e}^{t(\boldsymbol{z})}\Big[\prod_{a=1}^{D}(\boldsymbol{n}_a\cdot\boldsymbol{z}-q_a)^{m_a}\Big]P_{\boldsymbol{\alpha}_i}(\boldsymbol{z},t,\boldsymbol{m})$$

$$(\Psi_2)_{\boldsymbol{\alpha}_i}(\boldsymbol{z}) = \mathrm{e}^{t(\boldsymbol{z})}\Big[\prod_{a=1}^{D}(\boldsymbol{n}_a\cdot\boldsymbol{z}-q_a)^{m_a}\Big]\int_\Omega \frac{P_{\boldsymbol{\alpha}_i}(\boldsymbol{y},t)}{(z_1-y_1)\cdots(z_D-y_D)}\mathrm{d}\mu(\boldsymbol{y})$$

$$(\Psi_1^*)_{\boldsymbol{\alpha}_i}(\boldsymbol{z}) = \sum_{j=1}^{|[k]|}(H(t,\boldsymbol{m})^{-1})_{\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_j}\int_\Omega \frac{P_{\boldsymbol{\alpha}_j}(\boldsymbol{y},t,\boldsymbol{m})}{(z_1-y_1)\cdots(z_D-y_D)}\mathrm{d}\mu(\boldsymbol{y})$$

$$(\Psi_2^*)_{\boldsymbol{\alpha}_i}(\boldsymbol{z}) = \sum_{j=1}^{|[k]|}(H(t,\boldsymbol{m})^{-1})_{\boldsymbol{\alpha}_i,\boldsymbol{\alpha}_j}P_{\boldsymbol{\alpha}_j}(\boldsymbol{z},t,\boldsymbol{m})$$

The hierarchy

1. The Baker functions solve the linear system of differential equations

$$\frac{\partial \Psi}{\partial t_{\boldsymbol{\alpha}_j}} = \left(J_{\boldsymbol{\alpha}_j^{(\ell)}}\right)_+ \Psi, \qquad \frac{\partial \Psi^*}{\partial t_{\boldsymbol{\alpha}_j}} = -\left(J_{\boldsymbol{\alpha}_j^{(\ell)}}\right)_+^\top \Psi^*$$

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4. Zakharov–Shabat type equations:

$$\frac{\partial \big(J_{\boldsymbol{\alpha}_i^{(k)}}\big)_+}{\partial t_{\boldsymbol{\alpha}_j^{(\ell)}}} - \frac{\partial \big(J_{\boldsymbol{\alpha}_j^{(\ell)}}\big)_+}{\partial t_{\boldsymbol{\alpha}_i^{(k)}}} + \Big[\big(J_{\boldsymbol{\alpha}_i^{(k)}}\big)_+, \big(J_{\boldsymbol{\alpha}_j^{(\ell)}}\big)_+\Big] = 0$$

$$\frac{\partial \omega_a}{\partial t_{\boldsymbol{\alpha}}} - (T_a J_{\boldsymbol{\alpha}})_+ \omega_a + \omega_a (J_{\boldsymbol{\alpha}})_+ = 0$$

Bilinear equations

For any pair of times $(t, \boldsymbol{m})$ and $(t', \boldsymbol{m}')$, points $\boldsymbol{r}_1 \in \mathcal{D}^*_{\boldsymbol{\alpha}_j^{(\ell)}}(t', \boldsymbol{m}')$ and $\boldsymbol{r}_2 \in \mathcal{D}_{\boldsymbol{\alpha}_i^{(k)}}(t, \boldsymbol{m})$ in the respective domains of convergence and D-dimensional tori $\mathbb{T}^D(\boldsymbol{r}_1)$ and $\mathbb{T}^D(\boldsymbol{r}_2)$ (Shilov borders of polydisks) we can ensure that Baker and adjoint Baker functions satisfy the following bilinear identity

$$\int_{\mathbb{T}^D(\boldsymbol{r}_1)} (\Psi_1)_{\boldsymbol{\alpha}_i^{(k)}}(\boldsymbol{z}, t, \boldsymbol{m})(\Psi_1^*)_{\boldsymbol{\alpha}_j^{(\ell)}}(\boldsymbol{z}, t', \boldsymbol{m}')\mathrm{d}z_1 \cdots \mathrm{d}z_D = \int_{\mathbb{T}^D(\boldsymbol{r}_2)} (\Psi_2)_{\boldsymbol{\alpha}_i^{(k)}}(\boldsymbol{z}, t, \boldsymbol{m})(\Psi_2^*)_{\boldsymbol{\alpha}_j^{(\ell)}}(\boldsymbol{z}, t', \boldsymbol{m}')\mathrm{d}z_1 \cdots \mathrm{d}z_D$$

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Toda equations!

$$\frac{\partial}{\partial t_b}\Big(\frac{\partial H_{[k]}}{\partial t_a}H_{[k]}^{-1}\Big)=(\Lambda_a)_{[k],[k+1]}H_{[k+1]}\big[(\Lambda_b)_{[k],[k+1]}\big]^\top H_{[k]}^{-1} - H_{[k]}\big[(\Lambda_b)_{[k-1],[k]}\big]^\top H_{[k-1]}^{-1}(\Lambda_a)_{[k-1],[k]}$$

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$$\overbrace{\frac{\partial}{\partial t_b}\Big(\frac{\partial H_{[k]}}{\partial t_a}H_{[k]}^{-1}\Big)}^{\text{second partial difference}} = \overbrace{(\Lambda_a)_{[k],[k+1]}H_{[k+1]}[(\Lambda_b)_{[k],[k+1]}]^\top H_{[k]}^{-1}}^{\text{quadratic term in } H}$$

$$- \underbrace{H_{[k]}[(\Lambda_a)_{[k-1],[k]}]^\top H_{[k-1]}^{-1}(\Lambda_a)_{[k-1],[k]}}_{\text{quadratic term in } H}$$

The KP style

Linear systems

Baker functions Ψ_1, Ψ_2 are both solutions of

$$\boxed{\frac{\partial \Psi}{\partial \boldsymbol{n}_a} = T_a \Psi - (\Delta_a \beta)(\boldsymbol{n}_a \cdot \boldsymbol{\Lambda})\Psi}$$

$$\boxed{\frac{\partial \Psi}{\partial t_{(a,b)}} = \frac{\partial^2 \Psi}{\partial t_a \partial t_b} - (V_{a,b} + V_{b,a})\Psi}$$

$$\boxed{\begin{aligned}\frac{\partial \Psi}{\partial t_{(a,b,c)}} = \frac{\partial^3 \Psi}{\partial t_a \partial t_b \partial t_c} - V_{a,b}\frac{\partial \Psi}{\partial t_c} - V_{c,a}\frac{\partial \Psi}{\partial t_b} - V_{b,c}\frac{\partial \Psi}{\partial t_a} \\ - \Big(\frac{\partial V_{a,b}}{\partial t_c} + \frac{\partial V_{c,a}}{\partial t_b} + \frac{\partial V_{b,c}}{\partial t_a} + V_{(a,b,c)}\Big)\Psi\end{aligned}}$$

with $\frac{\partial}{\partial \boldsymbol{n}_a} := \boldsymbol{n}_a \cdot \nabla$, $V_{a,b} := \frac{\partial \beta}{\partial t_a}\Lambda_b$ and $V_{a,b,c} := \frac{\partial \beta^{(2)}}{\partial t_a}\Lambda_b\Lambda_c - \frac{\partial \beta}{\partial t_a}\Lambda_b \beta \Lambda_c$. Here β is the first subdiagonal of S and $\beta^{(2)}$ the second subdiagonal

- The previous linear systems are expressed in terms of the subdiagonals of S, which are the coefficients of the MOVPR and that can be expressed in terms of quasi-determinants of bordered truncated moment matrices. In particular β is important in the description of the hierachy and of the Toda equations

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- For the MOVPR the two first are

$$\frac{\partial P_{[k]}}{\partial \boldsymbol{n}_a} = (\boldsymbol{n}_a \cdot \boldsymbol{x} - q_a)\Delta_a P_{[k]} - q_a P_{[k]} - (\Delta_a \beta)_{[k]}(\boldsymbol{n}_a \cdot \boldsymbol{\Lambda})_{[k+1],[k]} P_{[k]}$$

$$\frac{\partial P_{[k]}}{\partial t_{(a,b)}}(\boldsymbol{x}) = \frac{\partial^2 P_{[k]}}{\partial t_a \partial t_b}(\boldsymbol{x}) + x_a \frac{\partial P_{[k]}}{\partial t_b}(\boldsymbol{x}) + x_b \frac{\partial P_{[k]}}{\partial t_a}(\boldsymbol{x}) - \Big(\frac{\partial \beta_{[k]}}{\partial t_a}(\Lambda_b)_{[k-1],[k]} + \frac{\partial \beta_{[k]}}{\partial t_b}(\Lambda_a)_{[k-1],[k]}\Big) P_{[k]}(\boldsymbol{x})$$


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Thank you
Thanks!!!