Sobolev biorthogonality and Gauss–Borel factorization

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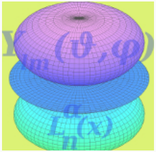

Prelimaries

Sobolev bilinear form

$$(f,h)_{\mathsf{Sobolev}} := \sum_{l,k=0}^{N} \int_{\Omega_{l,k}} f^{(l)}(x) h^{(k)}(x) \mathrm{d}\mu_{l,k}(x)$$

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Motivation: Lewis 1947

For a given $f(x)$ determine a polynomial $\Pi_k(x)$ of $\deg \leq k$ that minimizes

$$\sum_{\ell=0}^{k} \int_{\Omega_\ell} \left| f^{(\ell)}(x) - \Pi_k^{(\ell)}(x) \right|^2 \mathrm{d}\mu_\ell(x)$$

Preliminaries

- (1962-1973) Integration by parts period (Althammer, Schäfke and Wolf)
- (1990-2000) Coherent Pairs + Discrete case (Iserles, Koch, Nørsett and Sanz-Serna; Marcellán, Petronilho, Perez, Piñar; de Bruin, Meijer ...)
- (2000-*) Asymptotics, Generalizations: Matrix, several variables,...

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Sobolev matrix of measures $\mathcal{S}$

Given $N \in \mathbb{N}$ and finite Borel measures $\{\mu_{i,j}\}_{0 \le i,j \le N}$, $\operatorname{supp}(\mathrm{d}\mu_{i,j}) = \Omega_{i,j}$

$$\mathcal{S} := \begin{pmatrix} \mathrm{d}\mu_{\mathbf{0},\mathbf{0}} & \mathrm{d}\mu_{\mathbf{0},\mathbf{1}} & \dots & \mathrm{d}\mu_{\mathbf{0},N} & 0 & \dots \\ \mathrm{d}\mu_{\mathbf{1},\mathbf{0}} & \mathrm{d}\mu_{\mathbf{1},\mathbf{1}} & \dots & \mathrm{d}\mu_{\mathbf{1},N} & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ \mathrm{d}\mu_{N,\mathbf{0}} & \mathrm{d}\mu_{N,\mathbf{1}} & \dots & \mathrm{d}\mu_{N,N} & 0 & \dots \\ 0 & 0 & & 0 & 0 & \dots \\ \vdots & \vdots & & \vdots & & \ddots \end{pmatrix}$$

Preliminaries

Sobolev bilinear form

$(\cdot,\cdot)_{\mathcal{S}} : \mathbb{R}[x] \times \mathbb{R}[x] \longrightarrow \mathbb{R}$ associated with $\mathcal{S}$ is defined

$$(p,q)_{\mathcal{S}} := \sum_{l,k=0}^{N} \left\langle p^{(l)}, q^{(k)} \right\rangle_{l,k}$$

$$\left\langle p^{(l)}, q^{(k)} \right\rangle_{l,k} := \int_{\Omega_{l,k}} \frac{\mathrm{d}^l p}{\mathrm{d}x^l} \frac{\mathrm{d}^k q}{\mathrm{d}x^k} \mathrm{d}\mu_{l,k}(x)$$

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Comments:

- Condition $|(x^i, x^j)_{\mathcal{S}}| < \infty \ \forall i, j \in \mathbb{N}$
- The case $N \longrightarrow \infty$ is included

Preliminaries. Examples

- Standard case $N = 0$

$$(p,q)_{\mathcal{S}} = \int_{\Omega} p(x)q(x)\mathrm{d}\mu(x) \quad \leftrightarrow \quad \mathcal{S} = \begin{pmatrix} \mathrm{d}\mu(x) & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

- Diagonal case

$$(p,q)_{\mathcal{S}} = \sum_{k=0}^{N} \langle p^{(k)}, q^{(k)} \rangle_k \quad \longleftrightarrow \quad \mathcal{S} = \begin{pmatrix} \mathrm{d}\mu_0 & 0 & \dots & 0 & \dots & \dots \\ 0 & \mathrm{d}\mu_1 & \ddots & \vdots & \dots & \dots \\ \vdots & \ddots & \ddots & 0 & & \\ 0 & \dots & 0 & \mathrm{d}\mu_N & \ddots & \\ \vdots & & \vdots & \ddots & 0 & \ddots \\ \vdots & & \vdots & & \ddots & \ddots \end{pmatrix}$$

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Preliminaries. Examples

- Particular case with $N = 2$

$$(p,q)_{\mathcal{S}} = \int p(x)q(x)\omega(x)\mathrm{d}x + p'(a)q'(a) + \int p''(x)q'(x)\nu(x)\mathrm{d}x$$

$$\mathcal{S} = \begin{pmatrix} \omega(x)\mathrm{d}x & 0 & 0 & \dots \\ 0 & \delta(x-a) & 0 & \dots \\ 0 & \nu(x)\mathrm{d}x & 0 & \dots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

- General case

$$(p,q)_{\mathcal{S}} := \sum_{l,k=0}^{N} \langle p^{(l)}, q^{(k)} \rangle_{l,k}$$

$$= \int_{\Omega} (p, p', \dots, p^{(N)}) \begin{pmatrix} \mathrm{d}\mu_{0,0} & \mathrm{d}\mu_{0,1} & \dots & \mathrm{d}\mu_{0,N} \\ \mathrm{d}\mu_{1,0} & \mathrm{d}\mu_{1,1} & \dots & \mathrm{d}\mu_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{d}\mu_{N,1} & \mathrm{d}\mu_{N,2} & \dots & \mathrm{d}\mu_{N,N} \end{pmatrix} \begin{pmatrix} q \\ q' \\ \vdots \\ q^{(N)} \end{pmatrix}$$

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Gram matrix

Given a Sobolev bilinear form $(\cdot,\cdot)_{\mathcal{S}}$ the corresponding Gram will be

$$G_{\mathcal{S}} := \begin{pmatrix} (G_{\mathcal{S}})_{0,0} & (G_{\mathcal{S}})_{0,1} & \dots & (G_{\mathcal{S}})_{0,j} & \dots \\ (G_{\mathcal{S}})_{1,0} & (G_{\mathcal{S}})_{1,1} & \dots & (G_{\mathcal{S}})_{1,j} & \dots \\ \vdots & \vdots & & \vdots & \\ (G_{\mathcal{S}})_{j,0} & (G_{\mathcal{S}})_{j,1} & \dots & (G_{\mathcal{S}})_{j,j} & \dots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix},$$

$$(G_{\mathcal{S}})_{l,k} := (x^l, x^k)_{\mathcal{S}}$$

Exprwssion in terms of standard moment matrices

It can be written in terms of the *standard* moment matrices $g_{l,r}$ of the measures $\mathrm{d}\mu_{l,r}$

$$G_{\mathcal{S}} = \sum_{\ell,r=0}^{N} D^{l} g_{l,r} (D^{r})^{\top} \qquad D := \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Gauss–Borel factorization and Sobolev orthogonality

Gauss–Borel factorization

The factorization

$G_{\mathcal{S}}$ admits a Gaussian factorization iff all possible truncations are not singular, $\det\left(G_{\mathcal{S}}^{[k]}\right) \neq 0 \ \forall k = 1, 2, \ldots$; in such a case there exist two semi-infinite lower unitriangular matrices S_1, S_2 and a diagonal matrix $H = \operatorname{diag}(h_0, h_1, \ldots)$ such that

$$G_{\mathcal{S}} := S_1^{-1} H (S_2)^{-\top}$$

Sobolev orthogonality

Sobolev polynomial sequences

The monic Sobolev polynomial sequences associated with the LU-factorized moment matrix $G_{\mathcal{S}}$ are defined to be

$$P_1(x) := S_1\chi(x) := \begin{pmatrix} P_{1,0}(x) \\ P_{1,1}(x) \\ \vdots \\ P_{1,k}(x) \\ \vdots \end{pmatrix} \quad P_2(x) := S_2\chi(x) := \begin{pmatrix} P_{2,0}(x) \\ P_{2,1}(x) \\ \vdots \\ P_{2,k}(x) \\ \vdots \end{pmatrix}$$

where $\chi(x) := (1, x, x^2, \dots)^\top$.

The last quasi-determinant are, in this case (d is a scalar)

$$\Theta_* \left[\begin{array}{c|c} A & B \\ \hline C & d \end{array}\right] = \frac{\det\left(\begin{array}{c|c} A & B \\ \hline C & d \end{array}\right)}{\det A}$$

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Sobolev orthogonality

Determinantal expressions

The Sobolev sequences can be expressed by means of the following quasi-determinantal formulae

$$P_{1,k}(x) = \Theta_* \left[\begin{array}{ccc|c} & & & 1 \\ & G_{\mathcal{S}}^{[k]} & & x \\ & & & \vdots \\ & & & x^{k-1} \\ \hline (G_{\mathcal{S}})_{k,0} & \dots & (G_{\mathcal{S}})_{k,k-1} & x^k \end{array}\right]$$

$$P_{2,k}(x) = \Theta_* \left[\begin{array}{ccc|c} & & & 1 \\ & \left(G_{\mathcal{S}}^{\top}\right)^{[k]} & & x \\ & & & \vdots \\ & & & x^{k-1} \\ \hline (G_{\mathcal{S}}^{\top})_{k,0} & \dots & (G_{\mathcal{S}}^{\top})_{k,k-1} & x^k \end{array}\right]$$

Sobolev biorthogonal polynomial sequences (SBPS)

The Sobolev sequences P_1 and P_2 are biorthogonal

$$(P_{1,l}, P_{2,k})_{\mathcal{S}} := h_l \delta_{l,k}$$

with the further orthogonality properties

$$(P_{1,l}, x^k)_{\mathcal{S}} := \delta_{l,k} h_l \quad \forall k \leq l \quad \Longrightarrow \quad \sum_{i=0}^{l} \sum_{j=0}^{k} \left\langle P_{1,l}^{(i)}, \frac{\mathrm{d}^j x^l}{\mathrm{d} x^j} \right\rangle_{k,j} = \begin{cases} 0 & \forall k < l \\ h_l & k = l \end{cases}$$

$$(x^k, P_{2,l})_{\mathcal{S}} := h_r \delta_{r,l} \quad \forall k \leq l \quad \Longrightarrow \quad \sum_{i=0}^{k} \sum_{j=0}^{l} \left\langle \frac{\mathrm{d}^j x^l}{\mathrm{d} x^j}, P_{2,l}^{(i)} \right\rangle_{j,k} = \begin{cases} 0 & \forall k < l \\ h_l & k = l \end{cases}$$

Sobolev second kind functions

Sobolev second kind functions

For $y \notin \Omega$

$$C_{1,l}(y) := \int_\Omega \sum_{k=0}^{l} \sum_{j=0}^{N} P_{1,l}^{(k)}(x) \mathrm{d}\mu_{k,j} \left[\frac{\partial^j}{\partial x^j} \left(\frac{1}{y-x} \right) \right] = \left(P_{1,l}(x), \frac{1}{y-x} \right)_{\mathcal{S}}$$

$$C_{2,l}(y) := \int_\Omega \sum_{k=0}^{N} \sum_{j=0}^{l} \left[\frac{\partial^j}{\partial x^j} \left(\frac{1}{y-x} \right) \right] \mathrm{d}\mu_{j,k} P_{2,l}^{(k)}(x) = \left(\frac{1}{y-x}, P_{2,l}(x) \right)_{\mathcal{S}}$$

Sobolev second kind functions

Sobolev second kind functions and Gaussian factorization

The associated Sobolev second kind functions admit the following representation for all y such that $|y| > \max\{|x|, x \in \Omega\}$

$$C_1(y) = H(S_2)^{-\top}\chi^*(y) := \begin{pmatrix} C_{1,0}(y) \\ C_{1,1}(y) \\ \vdots \\ C_{1,k}(y) \\ \vdots \end{pmatrix} \quad C_2(y) = H(S_1)^{-\top}\chi^*(y) := \begin{pmatrix} C_{2,0}(y) \\ C_{2,1}(y) \\ \vdots \\ C_{2,k}(y) \\ \vdots \end{pmatrix}$$

with $\chi^*(x) := \frac{1}{x}\chi(\frac{1}{x})$

Transposing the Sobolev matrix of measures

A natural question is to establish the relation between the SBPS (and associated second kind functions) that arise from a given measure matrix $\mathscr{S}$ and the ones associated with its transposed $\mathscr{S}^\top$

Trasposing $\mathscr{S}$

Let $P_{\mathscr{S},a}$ and $C_{\mathscr{S},a}$ with $a = 1, 2$ denote the SBPS and second kind functions that arise from the measure matrix $\mathscr{S}$ and $P_{\mathscr{S}^\top,a}$ and $C_{\mathscr{S}^\top,a}$ the ones corresponding to $\mathscr{S}^\top$. Then, we have

$$P_{\mathscr{S},1} = P_{\mathscr{S}^\top,2} \qquad P_{\mathscr{S},2} = P_{\mathscr{S}^\top,1}$$

$$C_{\mathscr{S},1} = C_{\mathscr{S}^\top,2} \qquad C_{\mathscr{S},2} = C_{\mathscr{S}^\top,1}$$

If $\mathscr{S} = \mathscr{S}^\top$ then $P_{\mathscr{S},1} = P_{\mathscr{S},2}$ and $C_{\mathscr{S},1} = C_{\mathscr{S},2}$, and $G_{\mathscr{S}} = G_{\mathscr{S}}^\top$ and the LU factorization is a **Cholesky factorization**, $S_1 = S_2$

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\end{aligned}
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Sobolev Christoffel–Darboux kernels

The kernels (and ABC theorem)

- Christoffel–Darboux kernel

$$K^{[l]}(x,y) := \sum_{k=0}^{l-1} P_{2,k}(x) h_k^{-1} P_{1,k}(y) = [P_2(x)^\top]^{[l]} \left(H^{-1}\right)^{[l]} [P_1(y)]^{[l]}$$
$$= \left(\chi(x)^{[l]}\right)^\top \left(G^{[l]}\right)^{-1} \chi(y)^{[l]}$$

- Mixed 1st CD kernel

$$\mathcal{K}_1^{[l]}(x,y) := \sum_{k=0}^{l-1} C_{2k}(x) h_k^{-1} P_{1k}(y) = [C_2(x)^\top]^{[l]} \left(H^{-1}\right)^{[l]} [P_1(y)]^{[l]}$$

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Sobolev Christoffel–Darboux kernels

Reproducing property and projection

The CD Kernel still has the reproducing property

$$\left(K^{[l]}(x,z), K^{[l]}(z,y)\right)_{\mathcal{S}} = K^{[l]}(x,y)$$

and acts as a projector onto the basis of the SBPS

▸ Christoffel and Geronimus transformations

Additive perturbations

Additive perturbations of the Sobolev matrix of measures

Additive perturbation of Gram matrices

Suppose that our Gram matrix can be written as $\breve{G} = G + g$. Since we assume that G has an associated SBPS, then it must be LU-factorizable; at the same time, the requirement that the SBPS associated to $\breve{G}$ exists implies that the latter matrix should be LU-factorizable too

$$\breve{S}_1^{-1} \breve{H} \left(\breve{S}_2^{-1} \right)^{\top} = S_1^{-1} H \left(S_2^{-1} \right)^{\top} + g \qquad (1)$$

Additive perturbations of the Sobolev matrix of measures

Additive perturbation of Gram matrices. Notation

We introduce the matrices

$$A := S_1 g S_2^\top \qquad M_1 := \check{S}_1 S_1^{-1} \qquad M_2 := \check{S}_2 S_2^{-1}$$

Connection matrices

The matrices M_1, M_2 are the connection matrices between original and perturbed polynomials

$$M_1 P_1(x) = \check{P}_1(x) \qquad M_2 P_2(x) = \check{P}_2(x)$$

and provide an Gauss–Borel factorization of the matrix $H + A$

$$M_1^{-1} H \left(M_2^{-1}\right)^\top = H + A$$

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Determinantal expressions

The basis change from the old SBPS to the new one is given

$$\breve{P}_{1,k}(x) = \Theta_* \left[\begin{array}{cccc|c} & & & & P_{1,0}(x) \\ & & (H+A)^{[k]} & & P_{1,1}(x) \\ & & & & \vdots \\ & & & & P_{1,k-1}(x) \\ \hline (A)_{k,0} & (A)_{k,1} & \dots & (A)_{k,k-1} & P_{1,k}(x) \end{array}\right]$$

$$\breve{P}_{2,k}(x) = \Theta_* \left[\begin{array}{cccc|c} & & & & (A)_{0,k} \\ & & (H+A)^{[k]} & & (A)_{1,k} \\ & & & & \vdots \\ & & & & (A)_{k,k-1} \\ \hline P_{2,0}(x) & P_{2,1}(x) & \dots & P_{2,k-1}(x) & P_{2,k}(x) \end{array}\right]$$

$$\breve{h}_k = \Theta_* \left[\begin{array}{cccc|c} & & & & (A)_{0,k} \\ & & (H+A)^{[k]} & & (A)_{1,k} \\ & & & & \vdots \\ & & & & (A)_{k-1,k} \\ \hline (A)_{k,0} & (A)_{k,1} & \dots & (A)_{k,k-1} & (H+A)_{k,k} \end{array}\right]$$

Additive perturbations and applications

Getting more or less known results via Gauss–Borel factorization

- Sobolev orthogonality and classical orthogonal polynomials
- Coherent pairs and connection formulas
- Discrete Sobolev bilinear forms. Uvarov Perturbations

Sobolev orthogonality and classical orthogonal polynomials

It is a well known fact that classical orthogonal polynomials can be regarded as a very specific case of SOPS.

Classical weights

If we denote the classical measures by u_γ, where γ refers to the parameters that define them, they are

- Hermite: $u(x) = \mathrm{e}^{-x^2}, \;\; x \in \mathbb{R} \; (\gamma = \emptyset)$.
- Laguerre: $u_\alpha(x) = x^\alpha \, \mathrm{e}^{-x}, \;\; \alpha > -1, \;\; x \in \mathbb{R}_+ \; (\gamma = \{\alpha\})$
- Jacobi: $u_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta, \;\; \alpha, \beta > -1, \;\; x \in (-1,1)$ $(\gamma = \{\alpha, \beta\})$

Sobolev orthogonality and classical orthogonal polynomials

It is a well known fact that classical orthogonal polynomials can be regarded as a very specific case of SOPS.

Classical weights

If we denote the classical measures by u_γ, where γ refers to the parameters that define them, they are

- Hermite: $u(x) = \mathrm{e}^{-x^2}, \ x \in \mathbb{R} \ (\gamma = \varnothing)$.
- Laguerre: $u_\alpha(x) = x^\alpha \, \mathrm{e}^{-x}, \ \alpha > -1, \ x \in \mathbb{R}_+ \ (\gamma = \{\alpha\})$
- Jacobi: $u_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta, \ \alpha, \beta > -1, \ x \in (-1,1)$ $(\gamma = \{\alpha, \beta\})$

Sobolev orthogonality and classical orthogonal polynomials

Use $P_\gamma(x) = S_\gamma \chi(x)$ to denote the monic orthogonal polynomials $\{P_{\gamma,n}\}_n$ associated to each of them in terms of the Cholesky factorization matrices S_γ of the corresponding moment matrix g_γ

Pearson equation

$$p_2(x)\frac{\mathrm{d}u_\gamma}{\mathrm{d}x} = p_{1,\gamma}(x)u_\gamma \qquad\qquad p_2^k(x)u_\gamma = u_{\gamma+k}$$

where $\deg[p_2] \leq 2$ and $\deg[p_{1,\gamma}] = 1$

- Hermite $p_1 = -2x,\ p_2 = 1.$
- Laguerre $p_{1,\alpha} = (\alpha - x),\ p_2 = x.$
- Jacobi $p_{1,\alpha,\beta} = -[(\alpha-\beta) + (\alpha+\beta)x],\ p_2 = 1 - x^2.$

$$\Rightarrow P_{(\gamma+1),n}(x) = \frac{P'_{\gamma,n+1}(x)}{n+1} \Rightarrow \boxed{DS_{\gamma+1} = S_\gamma D^{-1}}$$

Sobolev orthogonality and classical orthogonal polynomials

SOPS from classical orthogonal polynomial sequences

The SBPS $\breve{P}_k$ and norms $\breve{h}_k$ for the following inner product

$$(f, h) = \int f(x)h(x)u_\gamma(x)\mathrm{d}x + \lambda \int f'(x)h'(x)u_{\gamma+1}(x)\mathrm{d}x \quad \lambda > 0$$

are given by

$$\breve{P}_k(x) = P_{\gamma,k}(x) \qquad \breve{h}_k = h_{\gamma,k} + \lambda k^2 h_{\gamma+1,k-1}$$

Proof:

$$A = \lambda S_\gamma D S_{\gamma+1}^{-1} H_{\gamma+1} \left(S_\gamma D S_{\gamma+1}^{-1}\right)^\top = \lambda D H_{\gamma+1} D^\top$$
$$= \lambda \begin{pmatrix} 0 & & & & & \\ & 1^2 h_{\gamma+1,0} & & & & \\ & & 2^2 h_{\gamma+1,1} & & & \\ & & & \ddots & & \\ & & & & k^2 h_{\gamma+1,k-1} & \\ & & & & & \ddots \end{pmatrix}$$

Back to additive perturbations applications

We are interested in obtaining the SBPS associated to the inner product

$$(f,h)_{\text{coherent}} := \int f(x)h(x)\mathrm{d}\mu_1(x) + \lambda \int f'(x)h'(x)\mathrm{d}\mu_2(x) \quad \lambda > 0$$

where $\mathrm{d}\mu_1(x)$ and $\mathrm{d}\mu_2(x)$ form a *coherent pair of measures*, i.e., if there exist some non zero constants $\{r_k\}_{k=1}^{\infty}$ such that the corresponding OPS, $\{P_k\}_{k=0}^{\infty}$ and $\{Q_k\}_{k=0}^{\infty}$, are linked by the sturcture equations

$$Q_k(x) = \frac{1}{k+1} P'_{k+1}(x) - \frac{r_k}{k} P'_k(x)$$

This inner product, in terms of moment matrices reads

$$\breve{G} = g_1 + \lambda D g_2 D^{\top}$$

and therefore can be studied from the additive perturbation approach.

Coherent pairs and connection formulas

Let us introduce some notation for the moment matrices, their factorization and corresponding OPS:

$$\begin{aligned} \mathrm{d}\mu_1(x) &\longrightarrow g_1 = S^{-1} H \left(S^{-1}\right)^T &&\longrightarrow P(x) = S\chi(x) \\ \mathrm{d}\mu_2(x) &\longrightarrow g_2 = Z^{-1} K \left(Z^{-1}\right)^T &&\longrightarrow Q(x) = Z\chi(x) \ . \end{aligned}$$

Consider it as an additive perturbation

$$A = \lambda \left(SDZ^{-1}\right) K \left(SDZ^{-1}\right)^\top$$

Coherent pairs and connection formulas

We introduce the lower matrix R^{-1}

$$SDZ^{-1} = \left(\begin{array}{c} \mathbf{0}^\top \\ \hline R^{-1} \end{array}\right) = \left(\begin{array}{cccc} 0 & 0 & 0 & \dots \\ \hline 1 & 0 & 0 & \dots \\ (R^{-1})_{1,0} & 2 & 0 & \dots \\ (R^{-1})_{2,0} & (R^{-1})_{2,1} & 3 & \ddots \\ \vdots & \vdots & \vdots & \end{array}\right)$$

So that

$$A^{[k]} = \left(\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0}^\top & \lambda \left(R^{[k-1]}\right)^{-1} K^{[k-1]} \left(R^{[k-1]}\right)^{-\top} \end{array}\right)$$

Coherent pairs and connection formulas

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So that

$$A^{[k]} = \left(\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0}^\top & \lambda \left(R^{[k-1]}\right)^{-1} K^{[k-1]} \left(R^{[k-1]}\right)^{-\top} \end{array}\right)$$

Then, we deduce that the new SOPS is given by

$$\breve{P}_k = P_k(x) - \lambda \left(\left(R^{-1} K \left(R^{-1} \right)^T \right)^{[k]}_{k-1,0} \quad \ldots \quad \left(R^{-1} K \left(R^{-1} \right)^T \right)^{[k]}_{k-1,k-2} \right)$$

$$\times \left[\left(R^{-1} K \left(R^{-1} \right)^T \right)^{[k-1]} + H^{[k-1]} \right]^{-1} \begin{pmatrix} P_1(x) \\ P_2(x) \\ \vdots \\ P_{k-1}(x) \end{pmatrix}$$

Coherent pairs and connection formulas

$$SDZ^{-1}Q(x) = P'(x) \Rightarrow R^{-1}\begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} P_1' \\ P_2' \\ P_3' \\ \vdots \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \end{pmatrix} = R\begin{pmatrix} P_1' \\ P_2' \\ P_3' \\ \vdots \end{pmatrix}$$

But, due to coherence property, we know that

$$R = \begin{pmatrix} 1 & & & & \\ -\frac{r_1}{1} & \frac{1}{2} & & & \\ & -\frac{r_2}{2} & \frac{1}{3} & & \\ & & -\frac{r_3}{3} & \ddots & \\ & & & \ddots & \end{pmatrix}$$

Coherent pairs and connection formulas

$$SDZ^{-1}Q(x) = P'(x) \Rightarrow R^{-1}\begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} P_1' \\ P_2' \\ P_3' \\ \vdots \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \end{pmatrix} = R\begin{pmatrix} P_1' \\ P_2' \\ P_3' \\ \vdots \end{pmatrix}$$

But, due to coherence property, we know that

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Coherent pairs and connection formulas

It is now easy to see that after introducing the matrices

$$r := \begin{pmatrix} 0 & & & & \\ r_1 & 0 & & & \\ & r_2 & 0 & & \\ & & r_3 & \ddots & \\ & & & \ddots & \end{pmatrix} \qquad N := \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & \ddots \end{pmatrix}$$

We can write

$$RN = \mathbb{I} - r \Longrightarrow R^{-1} = N\left(\mathbb{I} - r\right)^{-1} = N\left(\mathbb{I} + r + r^2 + \dots\right)$$

$$\left(R^{[k]}\right)^{-1} = N^{[k]}\left(\mathbb{I}^{[k]} + r^{[k]} + \dots + (r^{k-1})^{[k]}\right)$$

It is now easy to see that after introducing the matrices

$$r := \begin{pmatrix} 0 & & & & \\ r_1 & 0 & & & \\ & r_2 & 0 & & \\ & & r_3 & \ddots & \\ & & & \ddots & \end{pmatrix} \qquad N := \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & \ddots \end{pmatrix}$$

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$$RN = \mathbb{I} - r \implies R^{-1} = N\left(\mathbb{I} - r\right)^{-1} = N\left(\mathbb{I} + r + r^2 + \dots\right)$$

$$\left(R^{[k]}\right)^{-1} = N^{[k]}\left(\mathbb{I}^{[k]} + r^{[k]} + \dots + (r^{k-1})^{[k]}\right)$$

Coherent pairs and connection formulas

Therefore

$$\lambda\left(R^{-1}K\left(R^{-1}\right)^{T}\right)^{[k]} = \lambda N^{[k]}\left(\mathbb{I}^{[k]}+r^{[k]}+(r^{2})^{[k]}+\cdots+(r^{k-1})^{[k]}\right)K^{[k]} \times\left(\mathbb{I}^{[k]}+r^{[k]}+(r^{2})^{[k]}+\cdots+(r^{k-1})^{[k]}\right)^{\top}N^{[k]}$$

On the perturbed SOPS

The perturbed SOPS, $\check{P}_k$, depend only on the first $k-1$ parameters $\{r_1, r_2, \ldots, r_{k-1}\}$ that characterized the coherence and the norms of the original polynomials.

Coherent pairs and connection formulas

Therefore

$$\lambda \left(R^{-1}K\left(R^{-1}\right)^{T}\right)^{[k]}$$
$$= \lambda N^{[k]}\left(\mathbb{I}^{[k]} + r^{[k]} + (r^{2})^{[k]} + \cdots + (r^{k-1})^{[k]}\right)K^{[k]}$$
$$\times\left(\mathbb{I}^{[k]} + r^{[k]} + (r^{2})^{[k]} + \cdots + (r^{k-1})^{[k]}\right)^{\top}N^{[k]}$$

On the perturbed SOPS

The perturbed SOPS, $\check{P}_k$, depend only on the first $k-1$ parameters $\{r_1, r_2, \ldots, r_{k-1}\}$ that characterized the coherence and the norms of the original polynomials.

$k = 3$ connection formula

$$\lambda\left(R^{-1}K\left(R^{-1}\right)^T\right)^{[3]}$$
$$=\lambda\begin{pmatrix} K_0 & 2r_1K_0 & 3r_2r_1K_0 \\ 2r_1K_0 & 2^2(r_1^2K_0+K_1) & 2\cdot 3(r_1^2r_2K_0+r_2K_1) \\ 3r_2r_1K_0 & 2\cdot 3(r_1^2r_2K_0+r_2K_1) & 3^2(r_1^2r_2^2K_0+r_2^2K_1+K_2)\end{pmatrix}$$

which yields

$$\breve{P}_0 = P_0, \quad \breve{P}_1 = P_1 \quad \breve{P}_2 = P_2 - \lambda(2r_1K_0)[\lambda K_0 + H_1]^{-1}P_1$$
$$\breve{P}_3 = P_3 - \lambda\begin{pmatrix}3r_2r_1K_0 & 2\cdot 3(r_1^2r_2K_0+r_2K_1)\end{pmatrix}$$
$$\times\begin{pmatrix} K_0+H_1 & 2r_1K_0 \\ 2r_1K_0 & 2^2(r_1^2K_0+K_1)+H_2\end{pmatrix}^{-1}\begin{pmatrix}P_1\\P_2\end{pmatrix}$$

The previous connection formulas for the Sobolev polynomials are a consequence of the lower bidiagonal structure of R

A possible generalization of the notion of coherent pairs can be obtained by considering a block bidiagonal R

Generalizing coherent pairs

Block coherent pairs I

We say that $\{\mathrm{d}\mu_1, \mathrm{d}\mu_2\}$ form a $m \times m$ block coherent pair if their associated OPS are related as follows

$$\begin{pmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{m-1} \end{pmatrix} = (R_m)_{[0][0]} \begin{pmatrix} P'_1 \\ P'_2 \\ \vdots \\ P'_m \end{pmatrix}$$

$$\begin{pmatrix} Q_{km} \\ Q_{km+1} \\ \vdots \\ Q_{km+m-1} \end{pmatrix} = (R_m)_{[k][k-1]} \begin{pmatrix} P'_{(k-1)m+1} \\ P'_{(k-1)m+2} \\ \vdots \\ P'_{(k-1)m+m} \end{pmatrix} + (R_m)_{[k][k]} \begin{pmatrix} P'_{km+1} \\ P'_{km+2} \\ \vdots \\ P'_{km+m} \end{pmatrix} \quad \forall k \geq 1$$

Generalizing coherent pairs

Block coherent pairs II

where $(R_m)_{[k][k-1]}\,, (R_m)_{[k][k]}$ are $m \times m$ blocks and

$$(R_m)_{[k][k]} = \begin{pmatrix} \frac{1}{km+1} & & & \\ * & \frac{1}{km+2} & & \\ \vdots & \vdots & \ddots & \\ * & * & \dots & \frac{1}{(k+1)m} \end{pmatrix}$$

Note that the case $m = 1$ reproduces just the standard concept of coherent pairs that we treated before. The case $m = 2$ contains as a particular case the symmetrically coherent pairs since

$$\begin{pmatrix} Q_{2k} \\ Q_{2k+1} \end{pmatrix} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \begin{pmatrix} P'_{2k-1} \\ P'_{2k} \end{pmatrix} + \begin{pmatrix} \frac{1}{2k+1} & 0 \\ 0 & \frac{1}{2k+1} \end{pmatrix} \begin{pmatrix} P'_{2k+1} \\ P'_{2k+2} \end{pmatrix}$$

Now $m = 2$ and take $k = 2$

$$\left(R_2^{[2\cdot 2]}\right)^{-1} = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} \left[\mathbb{I}_{4\times 4} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ r_2 & 0 & 0 & 0 \\ 0 & r_3 & 0 & 0 \end{pmatrix} \right]$$

$$\Longrightarrow \lambda \left(R^{-1} K \left(R^{-1} \right)^T \right)^{[4]} = \lambda \begin{pmatrix} K_0 & 0 & 3K_0 r_2 & 0 \\ 0 & 4K_1 & 0 & 8K_1 r_3 \\ 3K_0 r_2 & 0 & 9(K_2 + K_0 r_2^2) & 0 \\ 0 & 8K_1 r_3 & 0 & 16(K_3 + K_1 r_3^2) \end{pmatrix}$$

whence we deduce $\breve{P}_0 = P_0, \breve{P}_1 = P_1, \breve{P}_2 = P_2$

$$\breve{P}_3 = P_3 - \lambda \begin{pmatrix} 3K_0 r_2 & 0 \end{pmatrix} \left[\begin{pmatrix} K_0 & 0 \\ 0 & 4K_1 \end{pmatrix} + \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} \right]^{-1} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

$$\breve{P}_4 = P_4 - \lambda \begin{pmatrix} 0 & 8K_1 r_3 & 0 \end{pmatrix}$$

$$\times \left[\begin{pmatrix} K_0 & 0 & 3K_0 r_2 \\ 0 & 4K_1 & 0 \\ 3K_0 r_2 & 0 & 9(K_2 + k_0 r_2^2) \end{pmatrix} + \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & & H_3 \end{pmatrix} \right]^{-1} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Back to additive perturbations applications

Adding a discrete Sobolev contribution

Given a set of nodes and their multiplicities $\{x_i, n_i, m_i\}_{i=1}^{s}$ let us study the following Sobolev bilinear function

$$(f,h)_{\breve{\mathcal{g}}} := (f,h)_{\mathcal{g}} + \sum_{i=1}^{s} \sum_{k=0}^{n_i-1} \sum_{j=0}^{m_i-1} \xi_{k,j}^{(i)} h^{(k)}(x_i) f^{(j)}(x_i)$$

$$\breve{G} = G + g$$

Discrete Sobolev bilinear forms: Uvarov perturbations

Jets

Given a function f we introduce the jet vectors

$$\mathcal{J}_1[f(x)] := \Big(f(x_1), \dots f^{(n_1-1)}(x_1), \dots, f'(x_s), \dots, f^{(n_s-1)}(x_s) \Big)$$

$$\mathcal{J}_2[f(x)] := \Big(f(x_1), \dots f^{(m_1-1)}(x_1), \dots, f'(x_s), \dots, f^{(m_s-1)}(x_s) \Big)$$

Coupling matrices

We consider the following $\sum_i n_i \times \sum_i m_i$ matrix

$$\Xi := \begin{pmatrix} \xi^{(1)} & & & \\ & \xi^{(2)} & & \\ & & \ddots & \\ & & & \xi^{(2)} \end{pmatrix} \quad \xi^{(i)} := \begin{pmatrix} \xi^{(i)}_{0,0} & \xi^{(i)}_{0,1} & \dots & \xi^{(i)}_{0,m_i-1} \\ \xi^{(i)}_{1,0} & & & \\ \vdots & & & \\ \xi^{(i)}_{n_i-1} & & & \xi^{(i)}_{n_i-1,m_i-1} \end{pmatrix}$$

Discrete Sobolev bilinear forms: Uvarov perturbations

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$$\mathcal{J}_2[f(x)] := \Big(f(x_1),\dots f^{(m_1-1)}(x_1),\dots, f'(x_s),\dots, f^{(m_s-1)}(x_s)\Big)$$

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$$\Xi := \begin{pmatrix} \xi^{(1)} & & & \\ & \xi^{(2)} & & \\ & & \ddots & \\ & & & \xi^{(2)} \end{pmatrix} \quad \xi^{(i)} := \begin{pmatrix} \xi^{(i)}_{0,0} & \xi^{(i)}_{0,1} & \dots & \xi^{(i)}_{0,m_i-1} \\ \xi^{(i)}_{1,0} & & & \\ \vdots & & & \\ \xi^{(i)}_{n_i-1} & & & \xi^{(i)}_{n_i-1,m_i-1} \end{pmatrix}$$

Discrete Sobolev bilinear forms: Uvarov perturbations

The matrix A

Given an additive perturbation of a discrete Sobolev type form, the matrix A can be written in terms of the original polynomials as

$$A^{[k]} = \mathcal{J}_1[P_1^{[k]}]\Xi\mathcal{J}_2[P_2^{[k]}]^\top$$

Proof:

$$g = \mathcal{J}_1[\chi]\Xi\mathcal{J}_2[\chi]^\top \qquad A^{[k]} = S_1^{[k]}g^{[k]}\left(S_2^{[k]}\right)^\top$$

$$S_1^{[k]}\mathcal{J}_1[\chi] = \mathcal{J}_1[P_1^{[k]}] \qquad S_2^{[k]}\mathcal{J}_{[}\chi] = \mathcal{J}_2[P_2^{[k]}]$$

Discrete Sobolev bilinear forms: Uvarov perturbations

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Proof:

$$g = \mathcal{J}_1[\chi]\Xi\mathcal{J}_2[\chi]^\top \qquad A^{[k]} = S_1^{[k]} g^{[k]} \left(S_2^{[k]}\right)^\top$$

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Discrete Sobolev bilinear forms: Uvarov perturbations

The CD matrix

Define the following $\sum_i n_i \times \sum_i m_i$ matrix whose entries are the derivatives of the CD kernel evaluated at the points $\{x_i\}$ up to $\{(n_i - 1), (m_i - 1)\}$ times.

$$\mathbb{K}^{[k]} := \left(\mathcal{J}_2[P_2^{[k]}]\right)^\top \left(H^{[k]}\right)^{-1} \left(\mathcal{J}_1[P_1^{[k]}]\right)$$

Discrete Sobolev bilinear forms: Uvarov perturbations

$$\mathbb{K}^{[k]} := \begin{pmatrix} \mathbb{K}^{[k]}_{[1][1]} & \mathbb{K}^{[k]}_{[1][2]} & \dots & \mathbb{K}^{[k]}_{[1][s]} \\ \mathbb{K}^{[k]}_{[2][1]} & \mathbb{K}^{[k]}_{[2][2]} & \dots & \mathbb{K}^{[k]}_{[2][s]} \\ \\ \mathbb{K}^{[k]}_{[s][1]} & \mathbb{K}^{[k]}_{[s][2]} & \dots & \mathbb{K}^{[k]}_{[s][s]} \end{pmatrix}$$

where $\mathbb{K}^{[k]}_{[i][j]}$ is the following matrix

$$\begin{pmatrix} \left(K^{[k]}(x_i,x_j)\right)^{(0,0)} & \left(K^{[k]}(x_i,x_j)\right)^{(0,1)} & \dots & \left(K^{[k]}(x_i,x_j)\right)^{(0,n_j-1)} \\ \left(K^{[k]}(x_i,x_j)\right)^{(1,0)} & \left(K^{[k]}(x_i,x_j)\right)^{(1,1)} & \dots & \left(K^{[k]}(x_i,x_j)\right)^{(1,n_j-1)} \\ \\ \left(K^{[k]}(x_i,x_j)\right)^{(m_i-1,0)} & \left(K^{[k]}(x_i,x_j)\right)^{(m_i-1,1)} & \dots & \left(K^{[k]}(x_i,x_j)\right)^{(m_i-1,n_j-1)} \end{pmatrix}$$

With $\left(K^{[k]}(x_i,x_j)\right)^{(t,d)} := \frac{\partial^{t+d} K^{[k]}(x,y)}{\partial x^t \partial y^d}\Big|_{(x,y)=(x_i,x_j)}$

Discrete Sobolev bilinear forms: Uvarov perturbations

Quasideterminantal formulas

The perturbed SBPS via adding a discrete part can be represented in terms of the following involving only the original SBPS.

$$\check{P}_{1,k}(x) = \Theta\begin{pmatrix} \mathbb{I} + \mathbb{K}^{[k]}\Xi & \mathcal{J}_2[K^{[k]}(\cdot, x)]^\top \\ \mathcal{J}_1[P_{1,k}]\Xi & P_{1,k}(x) \end{pmatrix}$$

$$\check{P}_{2,k}(x) = \Theta\begin{pmatrix} \mathbb{I} + \Xi\mathbb{K}^{[k]} & \Xi\mathcal{J}_2[P_{2,k}]^\top \\ \mathcal{J}_1[K^{[k]}(x, \cdot)] & P_{2,k}(x) \end{pmatrix}$$

Here the expression $\mathcal{J}_2[K^{[k]}(\cdot, x)]$ ($\mathcal{J}_1[K^{[k]}(x, \cdot)]$) stands for the action of the jet $\mathcal{J}_1$ (respectively $\mathcal{J}_2$), on the first (second) variable of K.

Alternative formulas

The perturbed SBPS via adding a discrete part can be represented in terms of the following involving only the original SBPS.

$$\check{P}_{1,k}(x)$$
$$= \begin{pmatrix} -\mathscr{J}_1[P_{1,k}]\Xi\left(\mathbb{I}+\mathbb{K}^{[k]}\Xi\right)^{-1}\left(\mathscr{J}_2\left[(P_2^{[k]})^\top\right]\right)^\top\left(H^{[k]}\right)^{-1} & 1 \end{pmatrix}\begin{pmatrix} P_1^{[k]}(x) \\ P_{1,k}(x) \end{pmatrix},$$

$$\check{P}_{2,k}(x)$$
$$= \begin{pmatrix} \left(P_2^{[k]}(x)\right)^\top & P_{2,k}(x) \end{pmatrix}\begin{pmatrix} -\left(H^{[k]}\right)^{-1}\mathscr{J}_1[P_1^{[k]}]\left(\mathbb{I}+\Xi\mathbb{K}^{[k]}\right)^{-1}\Xi\mathscr{J}_2[P_{2,k}]^\top \\ 1 \end{pmatrix}$$

Back to additive perturbations applications

Integration by parts

A weight Sobolev case

$$(p,q)_{\mathcal{S}} = \sum_{i,j} \int_a^b p^{(i)}(x) q^{(j)}(x) \omega_{i,j}(x) \mathrm{d}x$$

$$\mathcal{S} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \omega_{i-1,j-1} & \omega_{i-1,j} & \omega_{i-1,j+1} & \cdots \\ \cdots & \omega_{i,j-1} & \color{blue}{\omega_{i,j}} & \omega_{i,j+1} & \cdots \\ \cdots & \omega_{i+1,j-1} & \omega_{i+1,j} & \omega_{i+1,j+1} & \\ & \vdots & \vdots & & \ddots \end{pmatrix} \mathrm{d}x$$

Integration by parts

Use the **integration by parts** technique

$$\int_a^b p^{(i)}\omega_{i,j}q^{(j)}\mathrm{d}x$$

$$= \begin{cases} -\int_a^b p^{(i)}\omega'_{i,j}q^{(j-1)}\mathrm{d}x - \int_a^b p^{(i+1)}\omega_{i,j}q^{(j-1)}\mathrm{d}x + \left[p^{(i)}(x)\omega_{i,j}(x)q^{(j-1)}(x)\right]_a^b \\ \\ -\int_a^b p^{(i-1)}\omega'_{i,j}q^{(j)}\mathrm{d}x - \int_a^b p^{(i-1)}\omega_{i,j}q^{(j+1)}\mathrm{d}x + \left[p^{(i-1)}(x)\omega_{i,j}(x)q^{(j)}(x)\right]_a^b \end{cases}$$

Equivalence classes of matrices of measures

$$\mathcal{S} \to \mathcal{S}_1 = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & \omega_{i-1,j-1} & \omega_{i-1,j} & \omega_{i-1,j+1} & \dots \\ \dots & \omega_{i,j-1} - \omega'_{i,j} \leftarrow & 0 & \omega_{i,j+1} & \dots \\ \dots & \omega_{i+1,j-1} - \omega_{i,j} \swarrow & \omega_{i+1,j} & \omega_{i+1,j+1} & \\ & \vdots & \vdots & & \ddots \end{pmatrix} \mathrm{d}x$$

$$+ \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & 0 & 0 & 0 & \dots \\ \dots & \delta_a^b \omega_{i,j} & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & \\ & \vdots & \vdots & & \ddots \end{pmatrix} \mathrm{d}x$$

with $\delta_a^b = \delta(x-b) - \delta(x-a)$

Integration by parts

$$\mathcal{S} \to \mathcal{S}_2 = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \omega_{i-1,j-1} & \begin{matrix} \omega_{i-1,j} - {\color{blue}\omega'_{i,j}} \\ {\color{blue}\uparrow} \end{matrix} & \begin{matrix} \omega_{i-1,j+1} - {\color{blue}\omega_{i,j}} \\ {\color{blue}\nearrow} \end{matrix} & \cdots \\ \cdots & \omega_{i,j-1} & 0 & \omega_{i,j+1} & \cdots \\ \cdots & \omega_{i+1,j-1} & \omega_{i+1,j} & \omega_{i+1,j+1} & \\ & \vdots & \vdots & & \ddots \end{pmatrix} \mathrm{d}x$$

$$+ \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & 0 & {\color{blue}\delta_a^b \omega_{i,j}} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \\ & \vdots & \vdots & & \ddots \end{pmatrix} \mathrm{d}x$$

Integration by parts

Equivalence

All the three matrices $\mathscr{S}$, $\mathscr{S}_1$ and $\mathscr{S}_2$ give the same Sobolev bilinear form

$$(\cdot,\cdot)_{\mathscr{S}} = (\cdot,\cdot)_{\mathscr{S}_1} = (\cdot,\cdot)_{\mathscr{S}_2}$$

Equivalence classes

Equivalent Sobolev matrices of measures

- $\mathcal{S}_a \sim \mathcal{S}_b$ iff $(p,q)_{\mathcal{S}_a} = (p,q)_{\mathcal{S}_b}$ for every $p(x), q(x) \in \mathbb{R}[x]$
- **Equivalence** class $[\mathcal{S}_a] = \{\mathcal{S}_b : \mathcal{S}_b \sim \mathcal{S}_a\}$

Comments, if $\mathcal{S}_a \sim \mathcal{S}_b$

- Same Gram matrices $G_{\mathcal{S}_a} = G_{\mathcal{S}_b}$
- Same SBPS $\left(P_{\mathcal{S}_a}\right)_k = \left(P_{\mathcal{S}_b}\right)_k$ for $k \in \mathbb{N}$

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Symmetric Sobolev matrices of measures

If $\mathcal{S} = \mathcal{S}^{\top} \implies \mathcal{S} \sim$ Diagonal Sobolev + boundary terms

$$\begin{pmatrix} \omega_{0,0} & \omega_{1,0} & \omega_{2,0} & \omega_{3,0} \\ \omega_{1,0} & \omega_{1,1} & \omega_{2,1} & \omega_{3,1} \\ \omega_{2,0} & \omega_{2,1} & \omega_{2,2} & \omega_{3,2} \\ \omega_{3,0} & \omega_{3,1} & \omega_{3,2} & \omega_{3,3} \end{pmatrix} \sim \operatorname{diag} + \text{B.T.}$$

with

$$\begin{aligned} \operatorname{diag} := \ & \operatorname{diag}(\omega_{0,0} - \omega'_{1,0} + \omega''_{2,0} + \omega'''_{3,0}, \\ & \omega_{1,1} - \omega'_{2,1} + \omega''_{3,1} - 2\omega_{2,0} + 3\omega'_{3,0}, \omega_{2,2} - \omega'_{2,3} - 2\omega_{3,1}, \omega_{3,3}) \end{aligned}$$

$$\text{B.T.} := \begin{pmatrix} \delta[\omega_{1,0} - \omega'_{2,0} - \omega''_{3,0}] & \delta[\omega_{2,0} - \omega'_{3,0}] & \delta\omega_{3,0} & 0 \\ \delta[\omega_{2,0} - \omega'_{3,0}] & \delta[\omega_{2,1} - \omega_{3,0} - \omega'_{3,1}] & \delta\omega_{3,1} & 0 \\ \delta\omega_{3,0} & \delta\omega_{3,1} & \delta\omega_{3,2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Christoffel and Geronimus perturbations

Christoffel perturbations

The matrices $\Lambda, \mathcal{X}$

$$\Lambda := \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad \mathcal{X} := \begin{pmatrix} x & 1 & 0 & 0 & \dots \\ 0 & x & 2 & 0 & \dots \\ 0 & 0 & x & 3 & \dots \\ 0 & 0 & 0 & x & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Multiplication by x and the matrix $\mathcal{X}$

$$(xf, h)_{\mathcal{S}} = (f, h)_{\mathcal{X}\mathcal{S}} \qquad \Lambda G_{\mathcal{S}} = G_{\mathcal{X}\mathcal{S}}$$

$$(f, xh)_{\mathcal{S}} = (f, h)_{\mathcal{S}\mathcal{X}^\top} \qquad G_{\mathcal{S}}\Lambda^\top = G_{\mathcal{S}\mathcal{X}^\top}$$

Christoffel perturbations

The matrices $\Lambda, \mathcal{X}$

$$\Lambda := \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad \mathcal{X} := \begin{pmatrix} x & 1 & 0 & 0 & \dots \\ 0 & x & 2 & 0 & \dots \\ 0 & 0 & x & 3 & \dots \\ 0 & 0 & 0 & x & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Multiplication by x and the matrix $\mathcal{X}$

$$(xf, h)_{\mathscr{S}} = (f, h)_{\mathcal{X}\mathscr{S}} \qquad \Lambda G_{\mathscr{S}} = G_{\mathcal{X}\mathscr{S}}$$

$$(f, xh)_{\mathscr{S}} = (f, h)_{\mathscr{S}\mathcal{X}^{\top}} \qquad G_{\mathscr{S}}\Lambda^{\top} = G_{\mathscr{S}\mathcal{X}^{\top}}$$

Christoffel perturbations

Using the matrices $\Lambda, \mathcal{X}$

Given two real polynomials $P(x)$ and $Q(x)$, we have

$$(P(x)f, Q(x)h)_{\mathcal{S}} = (f, h)_{P(\mathcal{X})\mathcal{S}Q(\mathcal{X})^\top}$$
$$P(\Lambda)G_{\mathcal{S}}Q(\Lambda)^\top = G_{P(\mathcal{X})\mathcal{S}Q(\mathcal{X})^\top}$$

$P(\mathcal{X})$ is the following upper triangular matrix

$$P(\mathcal{X}) = \begin{pmatrix} P(x) & P'(x) & P''(x) & P'''(x) & \dots \\ 0 & P(x) & 2P'(x) & 3P''(x) & \dots \\ 0 & 0 & P(x) & 3P'(x) & \dots \\ 0 & 0 & 0 & P(x) & \dots \\ & & & & \ddots \\ & & & & \end{pmatrix}$$

Christoffel perturbations

Observations

If $\deg P = k$, then

$$(P(\mathfrak{X}))_{(n-1),(n-1)+i} = \begin{cases} \frac{(n)^i}{i!}\frac{\mathrm{d}^i P(x)}{\mathrm{d}x^i} & 0 \leq i \leq k \\ 0 & i > k \end{cases}$$

If $\mathcal{S}$ is a $(N+1)\times(N+1)$ measure matrix, then $P(\mathcal{X})\mathcal{S}\,Q(\mathcal{X})^\top$ will be a $(N+1)\times(N+1)$ measure matrix

Christoffel perturbations

The perturbation

Perturbing monic polynomial $R(x) := \prod_{i=1}^{d}(x - r_i)^{m_i}$ of degree $\sum_{i=1}^{d} m_i = M$ and perturbed Sobolev bilinear forms

$$(f, h)_{\hat{\mathcal{S}}_L} = (Rf, h)_{\mathcal{S}} \qquad (f, h)_{\hat{\mathcal{S}}_R} = (f, Rh)_{\mathcal{S}}$$

The right and left Christoffel–Sobolev deformed measure matrices and moment matrices are

$$\hat{\mathcal{S}}_L := R(\mathcal{X})\mathcal{S} \qquad \hat{\mathcal{S}}_R := \mathcal{S}[R(\mathcal{X})]^\top$$

$$R(\Lambda)G_{\mathcal{S}} = G_{\hat{\mathcal{S}}_L} := \hat{G}_L \qquad G_{\mathcal{S}}[R(\Lambda)]^\top = G_{\hat{\mathcal{S}}_R} := \hat{G}_R$$

Christoffel perturbations

Connectors

The connectors and adjoint connectors are defined as

$$\hat{\omega}_L := \hat{S}_{L1} R(\Lambda) S_1^{-1} \qquad \hat{\Omega}_L := S_2 \hat{S}_{L2}^{-1}$$
$$\hat{\omega}_R := \hat{S}_{R2} R(\Lambda) S_2^{-1} \qquad \hat{\Omega}_R := S_1 \hat{S}_{R1}^{-1}$$

Relations

The connectors are related to the adjoint connectors

$$\hat{\omega}_L = \hat{H}_L \hat{\Omega}_L^\top H^{-1} \qquad \hat{\omega}_R = \hat{H}_R \hat{\Omega}_R^\top H^{-1}$$

Christoffel perturbations

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The connectors are related to the adjoint connectors

$$\hat{\omega}_L = \hat{H}_L \hat{\Omega}_L^{\top} H^{-1} \qquad \hat{\omega}_R = \hat{H}_R \hat{\Omega}_R^{\top} H^{-1}$$

Christoffel perturbations

Band structure

$(M+1)$ band structure

$$\hat{\omega} = \begin{pmatrix} \hat{\omega}_{0,0} & \hat{\omega}_{0,1} & \dots & \hat{\omega}_{0,(M-1)} & \hat{\omega}_{0,M} & 0 & & & \\ 0 & \hat{\omega}_{1,1} & & & \hat{\omega}_{1,M} & \hat{\omega}_{1,(M+1)} & 0 & \dots & \\ 0 & 0 & \ddots & & & & \ddots & & \\ & & & \hat{\omega}_{k,k} & & & \hat{\omega}_{k,k+M-1} & \hat{\omega}_{k,k+M} & 0 \\ & & & & \ddots & & & & \ddots \end{pmatrix}$$

where $\hat{\omega}_{k,k+M} = 1$ and $\hat{\omega}_{k,k} = \frac{\hat{h}_k}{h_k}$

Connection formulas

Deformed and non deformed polynomials are related by the resolvents

$$\hat{\omega}_L P_1(x) = R(x)\hat{P}_{L1}(x) \qquad \hat{\Omega}_L \hat{P}_{L2}(x) = P_2(x)$$
$$\hat{\omega}_R P_2(x) = R(x)\hat{P}_{R2}(x) \qquad \hat{\Omega}_R \hat{P}_{R1}(x) = P_1(x)$$

Christoffel perturbations

Band structure

$(M+1)$ band structure

$$\hat{\omega} = \begin{pmatrix} \hat{\omega}_{0,0} & \hat{\omega}_{0,1} & \cdots & \hat{\omega}_{0,(M-1)} & \hat{\omega}_{0,M} & 0 & & & \\ 0 & \hat{\omega}_{1,1} & & & \hat{\omega}_{1,M} & \hat{\omega}_{1,(M+1)} & 0 & \cdots & \\ 0 & 0 & \ddots & & & & \ddots & & \\ & & & \hat{\omega}_{k,k} & & & \hat{\omega}_{k,k+M-1} & \hat{\omega}_{k,k+M} & 0 \\ & & & & \ddots & & & & \ddots \end{pmatrix}$$

where $\hat{\omega}_{k,k+M} = 1$ and $\hat{\omega}_{k,k} = \frac{\hat{h}_k}{h_k}$

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$$\hat{\omega}_L P_1(x) = R(x)\hat{P}_{L1}(x) \qquad \hat{\Omega}_L \hat{P}_{L2}(x) = P_2(x)$$

$$\hat{\omega}_R P_2(x) = R(x)\hat{P}_{R2}(x) \qquad \hat{\Omega}_R \hat{P}_{R1}(x) = P_1(x)$$

Christoffel perturbations

Transformed and non transformed Christoffel–Darboux kernel I

$$K^{[n+1]}(x,y) = R(y)\hat{K}_L^{[n+1]}(x,y) - \begin{pmatrix} (\hat{P}_{L2})_{n+1-M} & \dots & (\hat{P}_{L2})_n \end{pmatrix}$$

$$\times \begin{pmatrix} (\hat{h}_L)^{-1}_{n+1-M} & & \\ & \ddots & \\ & & (\hat{h}_L)^{-1}_n \end{pmatrix} \begin{pmatrix} (\hat{\omega}_L)_{n+1-M,n+1} & & 0 \\ \vdots & \ddots & \\ (\hat{\omega}_L)_{n,n+1} & \dots & (\hat{\omega}_L)_{n,n+M} \end{pmatrix}$$

$$\times \begin{pmatrix} (P_1)_{n+1}(y) \\ \vdots \\ (P_1)_{n+m}(y) \end{pmatrix}$$

Transformed and non transformed Christoffel–Darboux kernel II

$$K^{[n+1]}(y,x) = R(y)\hat{K}_R^{[n+1]}(y,x) - \begin{pmatrix} (\hat{P}_{R1})_{n+1-M} & \dots & (\hat{P}_{R1})_n \end{pmatrix}$$

$$\times \begin{pmatrix} (\hat{h}_R)^{-1}_{n+1-M} & & \\ & \ddots & \\ & & (\hat{h}_R)^{-1}_n \end{pmatrix} \begin{pmatrix} (\hat{\omega}_R)_{n+1-M,n+1} & & 0 \\ \vdots & \ddots & \\ (\hat{\omega}_R)_{n,n+1} & \dots & (\hat{\omega}_R)_{n,n+M} \end{pmatrix}$$

$$\times \begin{pmatrix} (P_2)_{n+1}(y) \\ \vdots \\ (P_2)_{n+m}(y) \end{pmatrix}$$

Christoffel perturbations

Jet

Given a function $f(x)$ we define the jet

$$\mathcal{J}_R[f] := \left(\frac{f^{(0)}(r_1)}{0!}, \dots, \frac{f^{(m_1-1)}(r_1)}{(m_1-1)!}; \dots; \frac{f^{(0)}(r_d)}{0!}, \dots, \frac{f^{(m_d-1)}(r_d)}{(m_d-1)!} \right)$$

Christoffel perturbations

Christoffel formulas for the left Christoffel transformation I

The norms are given in terms of the original ones by means of the relations

$$(\hat{h}_L)_n = \Theta_* \left[\begin{array}{c|c} \mathcal{J}_R \begin{bmatrix} (P_1)_n \\ (P_1)_{n+1} \\ \vdots \\ (P_1)_{n+M-1} \end{bmatrix} & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \mathcal{J}_R[(P_1)_{n+M}] & 0 \end{array}\right] h_n$$

Christoffel formulas for the left Christoffel transformation II

The Christoffel left transformed polynomials can expressed in terms of the original ones

$$(\hat{P}_{1L})_n(x) = \frac{1}{R(x)}\Theta_*\left[\begin{array}{c|c} \mathcal{J}_R\begin{bmatrix}(P_1)_n\\(P_1)_{n+1}\\\vdots\\(P_1)_{n+M-1}\end{bmatrix} & \begin{matrix}(P_1)_n(x)\\(P_1)_{n+1}(x)\\\vdots\\(P_1)_{n+M-1}(x)\end{matrix}\\\hline \mathcal{J}_R[(P_1)_{n+M}] & (P_1)_{n+M}(x)\end{array}\right]$$

$$\frac{(\hat{P}_{2L})_n(x)}{(\hat{h}_L)_n} = \Theta_*\left[\begin{array}{c|c} \mathcal{J}_R\begin{bmatrix}(P_1)_{n+1}\\\vdots\\(P_1)_{n+M}\end{bmatrix} & \begin{matrix}0\\\vdots\\1\end{matrix}\\\hline \mathcal{J}_R[K^{[n+1]}(x,\cdot)] & 0\end{array}\right]$$

Christoffel perturbations

Christoffel formulas for right Christoffel transformations I

The norms are given in terms of the original ones by means of the relations

$$(\hat{h}_R)_n = \Theta_* \left[\begin{array}{c|c} \mathcal{J}_R \begin{bmatrix} (P_2)_n \\ (P_2)_{n+1} \\ \vdots \\ (P_2)_{n+M-1} \end{bmatrix} & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \mathcal{J}_R[(P_2)_{n+M}] & 0 \end{array}\right] h_n$$

Christoffel formulas for right Christoffel transformations II

The Christoffel right transformed polynomials can expressed in terms of the original ones

$$(\hat{P}_{2R})_n(x) = \frac{1}{R(x)}\Theta_* \left[\begin{array}{c|c} \mathcal{J}_R\begin{bmatrix}(P_2)_n\\(P_2)_{n+1}\\\vdots\\(P_2)_{n+M-1}\end{bmatrix} & \begin{matrix}(P_2)_n(x)\\(P_2)_{n+1}(x)\\\vdots\\(P_2)_{n+M-1}(x)\end{matrix}\\\hline \mathcal{J}_R[(P_2)_{n+M}] & (P_2)_{n+M}(x)\end{array}\right]$$

$$\frac{(\hat{P}_{1R})_n(x)}{(\hat{h}_R)_n} = \Theta_* \left[\begin{array}{c|c} \mathcal{J}_R\begin{bmatrix}(P_2)_{n+1}\\\vdots\\(P_2)_{n+M}\end{bmatrix} & \begin{matrix}0\\\vdots\\1\end{matrix}\\\hline \mathcal{J}_R[K^{[n+1]}(\cdot,x)] & 0\end{array}\right]$$

The perturbation

Perturbing monic polynomial $Q(x) := \prod_{i=1}^{d}(x - q_i)^{m_i}$ of degree $\sum_{i=1}^{d} m_i = M$, $q_i \notin \operatorname{supp} \mathcal{S}$, and perturbed Sobolev bilinear forms

$$(Qf, h)_{\check{\mathcal{S}}_L} = (f, h)_{\mathcal{S}} \qquad (f, Qh)_{\check{\mathcal{S}}_R} = (f, h)_{\mathcal{S}}$$

The right and left Christoffel–Sobolev deformed measure matrices and moment matrices are

$$Q(\mathcal{X})\check{\mathcal{S}}_L = \mathcal{S} \qquad \check{\mathcal{S}}_R[Q(\mathcal{X})]^\top = \mathcal{S}$$

$$Q(\Lambda)G_{\check{\mathcal{S}}_L} = G_{\mathcal{S}} \qquad G_{\check{\mathcal{S}}_R}(Q(\Lambda))^\top = G_{\mathcal{S}}$$

Geronimus perturbations

The perturbation

$$\check{\mathcal{S}}_L := [Q(\mathcal{X})]^{-1}\,\mathcal{S} + \sum_{i=1}^{s} \xi^{(i)}\delta(x-q_i)\mathrm{d}x$$

$$\check{\mathcal{S}}_R := \mathcal{S}\left[Q(\mathcal{X}^\top)\right]^{-1} + \sum_{i=1}^{s} \xi^{(i)}\delta(x-q_i)\mathrm{d}x$$

$$\xi^{(i)} := \begin{pmatrix} \frac{\xi^{(i)}_{0,0}}{0!0!} & \frac{\xi^{(i)}_{0,1}}{0!1!} & \dots & \frac{\xi^{(i)}_{0,m_i-1}}{0!(m_i-1)!} & 0 & \dots \\ \frac{\xi^{(i)}_{1,0}}{1!0!} & \ddots & & & & \\ \vdots & & \ddots & & & \\ \frac{\xi^{(i)}_{m_i-1,0}}{(m_i-1)!0!} & & & \frac{\xi^{(i)}_{m_i-1,m_i-1}}{(m_i-1)!(m_i-1)!} & 0 & \dots \\ 0 & \dots & \dots & 0 & 0 & \\ \vdots & & & \vdots & \vdots & \ddots \end{pmatrix}$$

Geronimus perturbations

Connectors

$$\check{\omega}_L := \check{H}_L \check{S}_{1L}^{-\top} Q(\Lambda^\top) S_1^\top H^{-1} = \check{S}_{2L} S_2^{-1}$$
$$\check{\omega}_R := \check{H}_R \check{S}_{2R}^{-\top} Q(\Lambda^\top) S_2^\top H^{-1} = \check{S}_{1R} S_1^{-1}$$

Geronimus perturbations

Banded structure

$$\check{\omega} = \begin{pmatrix} \check{\omega}_{0,0} & 0 & \dots & & & & \\ \check{\omega}_{1,0} & \check{\omega}_{1,1} & \ddots & & & & \\ \vdots & \vdots & \ddots & & & & \\ \check{\omega}_{M,0} & \check{\omega}_{M,1} & & \check{\omega}_{M,M} & & & \\ 0 & \check{\omega}_{M+1,1} & & \check{\omega}_{M+1,M} & \check{\omega}_{M+1,M+1} & & \\ \vdots & \ddots & \ddots & & & \ddots & \\ & & & \check{\omega}_{k,k-M} & & & \check{\omega}_{k,k} \\ & & & & \ddots & & & \ddots \end{pmatrix}$$

where $\check{\omega}_{k,k-M} = \frac{\check{h}_k}{h_{k-M}}$ $\forall k > M$ and $\check{\omega}_{k,k} = 1$

Geronimus perturbations

$$
\mathcal{Q} := \begin{pmatrix}
Q_1 & Q_2 & Q_3 & \dots & Q_{M-1} & 1 & 0 & \dots \\
Q_2 & Q_3 & \dots & Q_{M-1} & 1 & 0 & \dots & \\
Q_3 & \dots & Q_{M-1} & 1 & 0 & \dots & & \\
\dots & Q_{M-1} & 1 & 0 & \dots & & & \\
Q_{M-1} & 1 & 0 & \dots & & & & \\
1 & 0 & \dots & & & & & \\
0 & \dots & & & & & &
\end{pmatrix}
$$

Geronimus perturbations

Connection formulas

The Geronimus deformed polynomials and the associated second kind functions are related to the non transformed ones according to

$$\check{\omega}_L P_2(x) = \check{P}_{2L}(x) \quad \Rightarrow \quad \check{\omega}_L C_2(x) = Q(x)\check{C}_{2L}(x) - \check{H}_L \check{S}_{1L}^{-\top} \boldsymbol{Q}\chi(x)$$

$$\check{\omega}_R P_1(x) = \check{P}_{1R}(x) \quad \Rightarrow \quad \check{\omega}_R C_1(x) = Q(x)\check{C}_{1R}(x) - \check{H}_R \check{S}_{2R}^{-\top} \boldsymbol{Q}\chi(x)$$

Transformed CD kernels. I

$$
\begin{aligned}
\check{K}_R^{[k]}(x,y) &= Q(x)K^{[k]}(x,y) \\
&\quad - \begin{pmatrix} (\check{P}_{2R})_k(x) & \dots & (\check{P}_{2R})_{k+M-1}(x) \end{pmatrix} \begin{pmatrix} (\check{h}_R)_k^{-1} & & \\ & \ddots & \\ & & (\check{h}_R)_{k+M-1}^{-1} \end{pmatrix} \\
&\quad \begin{pmatrix} (\check{\omega}_R)_{k,k-M} & \dots & (\check{\omega}_R)_{k,k-1} \\ & \ddots & \vdots \\ & & (\check{\omega}_R)_{k+M-1,k-1} \end{pmatrix} \begin{pmatrix} (P_1)_{k-M}(y) \\ (P_1)_{k+1-M}(y) \\ \vdots \\ (P_1)_{k-1}(y) \end{pmatrix}
\end{aligned}
$$

Geronimus perturbations

Transformed CD kernel. II

$$
\check{K}_L^{[k]}(x,y) = Q(y)K^{[k]}(x,y) - \begin{pmatrix} (\check{P}_{1L})_k(x) & \dots & (\check{P}_{1L})_{k+M-1}(x) \end{pmatrix} \begin{pmatrix} (\check{h}_L)_k^{-1} & & \\ & \ddots & \\ & & (\check{h}_L)_{k+M-1}^{-1} \end{pmatrix} \begin{pmatrix} (\check{\omega}_L)_{k,k-M} & \dots & (\check{\omega}_L)_{k,k-1} \\ & \ddots & \vdots \\ & & (\check{\omega}_L)_{k+M-1,k-1} \end{pmatrix} \begin{pmatrix} (P_2)_{k-M}(x) \\ (P_2)_{k+1-M}(x) \\ \vdots \\ (P_2)_{k-1}(x) \end{pmatrix}
$$

Transformed mixed kernels $\forall k \geq M$. I

$$Q(x)\mathfrak{K}_2^{[k]}(x,y) - \begin{pmatrix} (\check{P}_{2R})_k(x) & \dots & (\check{P}_{2R})_{k+M-1}(x) \end{pmatrix} \begin{pmatrix} (\check{h}_R)_k^{-1} & & \\ & \ddots & \\ & & (\check{h}_R)_{k+M-1}^{-1} \end{pmatrix}$$

$$\begin{pmatrix} (\check{\omega}_R)_{k,k-M} & \dots & (\check{\omega}_R)_{k,k-1} \\ & \ddots & \\ & & (\check{\omega}_R)_{k+M-1,k-1} \end{pmatrix} \begin{pmatrix} (C_1)_{k-M}(y) \\ (C_1)_{k+1-M}(y) \\ \vdots \\ (C_1)_{k-1}(y) \end{pmatrix}$$

$$= Q(y)\check{\mathfrak{K}}_{2R}^{[k]}(x,y) - \left(\chi^{[M]}(x)\right)^{\top} \mathbf{Q}\chi^{[M]}(y)$$

Transformed mixed kernels $\forall k \geq M$. II

$$Q(y)\mathcal{K}_1^{[k]}(x,y) - \begin{pmatrix}(\check{P}_{1L})_k(y) & \dots & (\check{P}_{1L})_{k+M-1}(y)\end{pmatrix} \begin{pmatrix}(\check{h}_L)_k^{-1} & & \\ & \ddots & \\ & & (\check{h}_L)_{k+M-1}^{-1}\end{pmatrix} \begin{pmatrix}(\check{\omega}_L)_{k,k-M} & \dots & (\check{\omega}_L)_{k,k-1} \\ & \ddots & \\ & & (\check{\omega}_L)_{k+M-1,k-1}\end{pmatrix} \begin{pmatrix}(C_2)_{k-N}(x) \\ (C_2)_{k+1-M}(x) \\ \vdots \\ (C_2)_{k-1}(x)\end{pmatrix}$$

$$= Q(x)\check{\mathcal{K}}_{1L}^{[k]}(x,y) - \left(\chi^{[M]}(y)\right)^{\top} \mathbf{Q}\chi^{[M]}(x)$$

Geronimus perturbations

$$Q_i(x) := \frac{Q(x)}{(x-q_i)^{m_i}}$$

$$\Xi_{Rj} := \begin{pmatrix} \xi^{(i)}_{0,m_i-1} & \xi^{(i)}_{0,m_i-2} & \cdots & \xi^{(i)}_{0,0} \\ \xi^{(i)}_{1,m_i-1} & \ddots & & \\ \vdots & & \ddots & \\ \xi^{(i)}_{m_i-1,m_i-1} & & & \xi^{(i)}_{m_i-1,0} \end{pmatrix} \begin{pmatrix} \frac{Q_j^{(0)}(q_j)}{0!} & \frac{Q_j^{(1)}(q_j)}{1!} & \cdots & \frac{Q_j^{(m_j-2)}(q_j)}{(m_j-2)!} & \frac{Q_j^{(m_j-1)}(q_j)}{(m_j-1)!} \\ & \frac{Q_j^{(0)}(q_j)}{0!} & & & \frac{Q_j^{(m_j-2)}(q_j)}{(m_j-2)!} \\ & & \ddots & & \vdots \\ & & & \ddots & \frac{Q_j^{(1)}(q_j)}{1!} \\ & & & & \frac{Q_j^{(0)}(q_j)}{0!} \end{pmatrix}$$

Geronimus perturbations

$$\Xi_{Lj} := \begin{pmatrix} \xi^{(i)}_{m_i-1,0} & \xi^{(i)}_{m_i-2,0} & \cdots & \xi^{(i)}_{0,0} \\ \xi^{(i)}_{m_i-1,1} & \ddots & & \\ \vdots & & \ddots & \\ \xi^{(i)}_{m_i-1,m_i-1} & & & \xi^{(i)}_{0,m_i-1} \end{pmatrix}$$

$$\begin{pmatrix} \frac{Q_j^{(0)}(q_j)}{0!} & \frac{Q_j^{(1)}(q_j)}{1!} & \cdots & \frac{Q_j^{(m_j-2)}(q_j)}{(n_j-2)!} & \frac{Q_j^{(m_j-1)}(q_j)}{(m_j-1)!} \\ & \frac{Q_j^{(0)}(q_j)}{0!} & & & \frac{Q_j^{(m_j-2)}(q_j)}{(m_j-2)!} \\ & & \ddots & & \vdots \\ & & & \ddots & \frac{Q_j^{(1)}(q_j)}{1!} \\ & & & & \frac{Q_j^{(0)}(q_j)}{0!} \end{pmatrix}$$

Geronimus perturbations

$$\Xi_L := \begin{pmatrix} \Xi_{L1} & 0 & \dots & 0 \\ 0 & \Xi_{L2} & 0 & \\ & & \ddots & \\ & & & \Xi_{Ls} \end{pmatrix}, \quad \Xi_R := \begin{pmatrix} \Xi_{R1} & 0 & \dots & 0 \\ 0 & \Xi_{R2} & 0 & \\ & & \ddots & \\ & & & \Xi_{Rs} \end{pmatrix}$$

Christoffel formulas for the Geronimus perturbations I

For $k \geq M$

$$(\check{h}_R)_k = h_{k-N}\Theta_* \left[\begin{array}{c|c} \mathcal{J}_Q \begin{bmatrix} (C_1)_{k-M} \\ \vdots \\ (C_1)_{k-1} \end{bmatrix} - \mathcal{J}_Q \begin{bmatrix} (P_1)_{k-M} \\ \vdots \\ (P_1)_{k-1} \end{bmatrix} \Xi_R & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \mathcal{J}_Q[(C_1)_k] - \mathcal{J}_Q[(P_1)_k]\Xi_R & 0 \end{array}\right]$$

$$\Xi_L := \begin{pmatrix} \Xi_{L1} & 0 & \dots & 0 \\ 0 & \Xi_{L2} & 0 & \\ & & \ddots & \\ & & & \Xi_{Ls} \end{pmatrix}, \quad \Xi_R := \begin{pmatrix} \Xi_{R1} & 0 & \dots & 0 \\ 0 & \Xi_{R2} & 0 & \\ & & \ddots & \\ & & & \Xi_{Rs} \end{pmatrix}$$

Christoffel formulas for the Geronimus perturbations I

For $k \geq M$

$$(\check{h}_R)_k = h_{k-N}\Theta_* \left[\begin{array}{c|c} \mathcal{J}_Q \begin{bmatrix} (C_1)_{k-M} \\ \vdots \\ (C_1)_{k-1} \end{bmatrix} - \mathcal{J}_Q \begin{bmatrix} (P_1)_{k-M} \\ \vdots \\ (P_1)_{k-1} \end{bmatrix} \Xi_R & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \mathcal{J}_Q[(C_1)_k] - \mathcal{J}_Q[(P_1)_k]\Xi_R & 0 \end{array}\right]$$

Christoffel formulas for the Geronimus perturbations II

For $k \geq M$

$$(\check{P}_{1R})_k = \Theta_* \left[\begin{array}{c|c} \mathcal{J}_Q \begin{bmatrix} (C_1)_{k-M} \\ \vdots \\ (C_1)_{k-1} \end{bmatrix} - \mathcal{J}_Q \begin{bmatrix} (P_1)_{k-M} \\ \vdots \\ (P_1)_{k-1} \end{bmatrix} \Xi_R & \begin{matrix} (P_1)_{k-M} \\ \vdots \\ (P_1)_{k-1} \end{matrix} \\ \hline \mathcal{J}_Q[(C_1)_k] - \mathcal{J}_Q[(P_1)_k]\Xi_R & (P_1)_k(x) \end{array}\right]$$

$$\frac{(\check{P}_{2R})_k(x)}{(\check{h}_R)_k} = \Theta_* \left[\begin{array}{c|c} \mathcal{J}_Q \begin{bmatrix} (C_1)_{k-M} \\ \vdots \\ (C_1)_{k-1} \end{bmatrix} - \mathcal{J}_Q \begin{bmatrix} (P_1)_{k-M} \\ \vdots \\ (P_1)_{k-1} \end{bmatrix} \Xi_R & \begin{matrix} 1 \\ \vdots \\ 0 \end{matrix} \\ \hline Q(x)\left(\mathcal{J}_Q[\mathcal{K}_2^{[k]}(x,\cdot)] - \mathcal{J}_Q[K^{[k]}(x,\cdot)]\Xi_R\right) & 0 \\ \qquad + (\chi^{[M]}(x))^\top \boldsymbol{Q} \mathcal{J}_Q[\chi^{[M]}] & \end{array}\right]$$

Geronimus perturbations

Christoffel formulas for the Geronimus perturbations III

For $k \geq M$

$$\check{h}_{Lk} = h_{k-M}\Theta_* \left[\begin{array}{c|c} \mathscr{J}_Q \begin{bmatrix} (C_2)_{k-M} \\ \vdots \\ (C_2)_{k-1} \end{bmatrix} - \mathscr{J}_Q \begin{bmatrix} (P_2)_{k-M} \\ \vdots \\ (P_2)_{k-1} \end{bmatrix} \Xi_L & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \mathscr{J}_Q[(C_2)_k] - \mathscr{J}_Q[(P_2)_k]\Xi_L & 0 \end{array}\right]$$

Christoffel formulas for the Geronimus perturbations IV

For $k \geq M$

$$(\check{P}_{2L})_k = \Theta_* \left[\begin{array}{c|c} \mathcal{J}_Q \begin{bmatrix} (C_2)_{k-M} \\ \vdots \\ (C_2)_{k-1} \end{bmatrix} - \mathcal{J}_Q \begin{bmatrix} (P_2)_{k-M} \\ \vdots \\ (P_2)_{k-1} \end{bmatrix} \Xi_L & \begin{matrix} (P_2)_{k-M} \\ \vdots \\ (P_2)_{k-1} \end{matrix} \\ \hline \mathcal{J}_Q[(C_2)_k] - \mathcal{J}_Q[(P_2)_k]\Xi_L & (P_2)_k(x) \end{array}\right]$$

$$\frac{(\check{P}_{1L})_k(x)}{(\check{h}_L)_k} = \Theta_* \left[\begin{array}{c|c} \mathcal{J}_Q \begin{bmatrix} (C_2)_{k-M} \\ \vdots \\ (C_2)_{k-1} \end{bmatrix} - \mathcal{J}_Q \begin{bmatrix} (P_2)_{k-M} \\ \vdots \\ (P_2)_{k-1} \end{bmatrix} \Xi_L & \begin{matrix} 1 \\ \vdots \\ 0 \end{matrix} \\ \hline Q(x)\Big(\mathcal{J}_Q[\mathcal{K}_1^{[k]}(\cdot,x)] - \mathcal{J}_Q[K^{[k]}(\cdot,x)]\Xi_L\Big) & 0 \\ \qquad + (\chi^{[M]}(x))^\top \boldsymbol{Q}\mathcal{J}_Q[\chi^{[M]}] & \end{array}\right]$$

Composing Geronimus and Christoffel

The Sobolev linear spectral deformed measure matrices are defined to be the composition of both a Geronimus and Christoffel transformation

$$\tilde{\mathcal{S}}_{RL} := \widehat{(\check{\mathcal{S}}_R)}_L = R(\mathcal{X})\mathcal{S}\left[Q(\mathcal{X}^\top)\right]^{-1} + \sum_{i=1}^{s} R(\mathcal{X})\xi^{(i)}\delta(x-q_i)$$

$$\tilde{\mathcal{S}}_{LR} := \widehat{(\check{\mathcal{S}}_L)}_R = [Q(\mathcal{X})]^{-1}\,\mathcal{S}\,R(\mathcal{X}^\top) + \sum_{i=1}^{s} \xi^{(i)}R(\mathcal{X}^\top)\delta(x-q_i)$$

Composing Geronimus and Christoffel II

$$
\begin{aligned}
(f, Qh)_{\tilde{\mathscr{S}}_{RL}} &= (Rf, h)_{\mathscr{S}}, & (Qf, h)_{\tilde{\mathscr{S}}_{LR}} &= (f, Rh)_{\mathscr{S}} \\
\tilde{\mathscr{S}}_{RL} Q(\mathcal{X}^\top) &= R(\mathcal{X})\mathscr{S}, & Q(\mathcal{X})\tilde{\mathscr{S}}_{RL} &= \mathscr{S} R(\mathcal{X}^\top) \\
R(\Lambda) G_{\mathscr{S}} &= G_{\tilde{\mathscr{S}}_{RL}} Q(\Lambda^\top) & Q(\Lambda) G_{\mathscr{S}} &= G_{\tilde{\mathscr{S}}_{LR}} R(\Lambda^\top)
\end{aligned}
$$

$$
(\tilde{P}_{1RL})_k(x) = \frac{1}{R(x)}
$$

$$
\times\Theta_* \left[\begin{array}{c|c}
\mathscr{J}_R\begin{bmatrix}(P_1)_{k-N}\\ \vdots \\ (P_1)_{k+M-1}\end{bmatrix}, \mathscr{J}_Q\begin{bmatrix}(C_1)_{k-N}\\ \vdots \\ (C_1)_{k+M-1}\end{bmatrix} - \mathscr{J}_Q\begin{bmatrix}(P_1)_{k-N}\\ \vdots \\ (P_1)_{k+M-1}\end{bmatrix}\Xi_R & \begin{matrix}(P_1)_{k-N}\\ \vdots \\ (P_1)_{k+M-1}\end{matrix} \\
\hline
\mathscr{J}_R[(P_1)_{k+M}], \mathscr{J}_R[(C_1)_{k+M}] - \mathscr{J}_R[(P_1)_{k+M}]\Xi_R & (P_1)_{k+M}(x)
\end{array}\right]
$$

Linear differential operators

Linear differential operators and Sobolev bilinear forms

The relations

$$(f',h)_{\mathcal{S}} = (f,h)_{\Lambda^\top \mathcal{S}} \qquad DG_{\mathcal{S}} = G_{\Lambda^\top \mathcal{S}}$$
$$(f,h')_{\mathcal{S}} = (f,h)_{\mathcal{S}\Lambda} \qquad G_{\mathcal{S}} D^\top = G_{\mathcal{S}\Lambda}$$

hold

By linearity, we deduce that given any linear differential operator $\boldsymbol{L} := \sum_{n,m=0}^{\infty} a_{n,m} x^n \frac{\mathrm{d}^m}{\mathrm{d}x^m}$, acting on one of the entries of our inner product, we can translate its action into a matrix multiplying the initial moment matrix $L := \sum_{n,r=0}^{\infty} a_{n,r} D^r \Lambda^n$ or into a matrix multiplying the initial measure matrix $\mathcal{L} = \sum_{n,r=0}^{\infty} a_{n,r} (\Lambda^\top)^r \mathcal{X}^n$

Linear differential operators and Sobolev bilinear forms

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Linear differential operators and Sobolev bilinear forms

$$(\boldsymbol{L}_1[f], \boldsymbol{L}_2[h])_{\mathcal{S}} = (f, h)_{\mathfrak{L}_1 \mathcal{S} \mathfrak{L}_2^\top}$$
$$L_1 G_{\mathcal{S}} (L_2)^\top = G_{\mathfrak{L}_1 \mathcal{S} (\mathfrak{L}_2)^\top}$$

Matrix of weights

We will assume that the Sobolev matrix has the form

$$\mathcal{S} = \mathcal{W} \mathrm{d}\mu(x)$$

for a suitable matrix of weights and a given measure μ.

Linear differential operators and Sobolev bilinear forms

$$(\boldsymbol{L}_1[f], \boldsymbol{L}_2[h])_{\mathcal{S}} = (f, h)_{\mathfrak{L}_1 \mathcal{S} \mathfrak{L}_2^\top}$$
$$L_1 G_{\mathcal{S}} (L_2)^\top = G_{\mathfrak{L}_1 \mathcal{S} (\mathfrak{L}_2)^\top}$$

Matrix of weights

We will assume that the Sobolev matrix has the form

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for a suitable matrix of weights and a given measure μ.

Linear differential operators and matrices of weights

Suppose that a $(N+1)\times(N+1)$ matrix of weights satisfying $\det \mathcal{W}^{[k]}(x) \neq 0\ \forall x \in \operatorname{supp} \mathcal{W}$ and $k = 0, 1 \ldots, N$ is given, then the Sobolev bilinear function $(f, h)_{\mathcal{S}}$ is equivalent to a generalized diagonal Sobolev bilinear function

$$(f, h)_S := \sum_{k=0}^{N} \langle \boldsymbol{L}_k[f], \boldsymbol{U}_k[h] \rangle_{w_k \mathrm{d}\mu}$$

for suitable

$$\boldsymbol{L}_k = \frac{\mathrm{d}^k}{\mathrm{d}x^k} + \sum_{j=k+1} l_{jk}(x) \frac{\mathrm{d}^j}{\mathrm{d}x^j}, \quad \boldsymbol{U}_k = \frac{\mathrm{d}^k}{\mathrm{d}x^k} + \sum_{j=k+1} u_{kj}(x) \frac{\mathrm{d}^j}{\mathrm{d}x^j}$$

and a set of weights $\{w_k(x)\}_{k=0}^N$

Linear differential operators

The pair $S = \{\{\boldsymbol{L}_k\}, \{\boldsymbol{U}_k\}\}_{k=0}^{N}$ with

$$\boldsymbol{L}_k = \frac{\mathrm{d}^k}{\mathrm{d}x^k} + \sum_{j=k+1} l_{jk}(x)\frac{\mathrm{d}^j}{\mathrm{d}x^j} \qquad \boldsymbol{U}_k = \frac{\mathrm{d}^k}{\mathrm{d}x^k} + \sum_{j=k+1} u_{kj}(x)\frac{\mathrm{d}^j}{\mathrm{d}x^j}$$

is determined by the LU factorization of $\mathcal{W}$ by means of the relations

$$\mathcal{W}(x) = \begin{pmatrix} 1 & & & & \\ l_{1,0}(x) & 1 & & & \\ l_{2,0}(x) & l_{2,1}(x) & 1 & & \\ \vdots & \vdots & & \ddots & \\ & & & & \\ l_{N,0}(x) & l_{N,1}(x) & & & 1 \end{pmatrix} \operatorname{diag}(w_0, \dots, w_N)$$

$$\times \begin{pmatrix} 1 & u_{0,1}(x) & u_{0,2}(x) & \dots & u_{0,N}(x) \\ & 1 & u_{1,2}(x) & \dots & u_{1,N}(x) \\ & & 1 & & \\ & & & \ddots & \\ & & & & u_{N-1,N}(x) \\ & & & & 1 \end{pmatrix}$$

In addition, if each weight $w_k(x)$ is positive definite and $l_{j,k}(x), u_{k,j}(x)$ are polynomials satisfying the relations

$$j - \deg[u_{k,j}(x)] > k \qquad \text{and} \qquad j - \deg[l_{j,k}(x)] > k$$

then $G_{\mathcal{W}}$ is LU-factorizable and has a SBPS

Sobolev biorthogonality and integrable systems

Continuous ad commuting deformations the Gram matrix

We define the time-deformed moment matrix

$$G^{t}_{\mathcal{S}} = W^{t_1}_{1,0} G_{\mathcal{S}} [W^{t_2}_{2,0}]^{-1}$$

where the deformation matrices $W_{1,0}(t_1)$ and $W_{1,0}(t_2)$ are given by

$$W^{t_1}_{1,0} = \exp\left(\sum_{j=0}^{\infty} t_{1,j}\,\Lambda^{j}\right) \quad W^{t_2}_{2,0} = \exp\left(\sum_{j=0}^{\infty} t_{2,j}\left(\Lambda^{\top}\right)^{j}\right)$$

The flows for the Sobolev matrix

The deformed moment matrix $G^t_{\mathcal{S}}$ can be written as the Sobolev matrix associated to a time dependent measure matrix, this is

$$G^t_{\mathcal{S}} = G_{\mathcal{S}^t}$$

where the new time dependent matrix off measures is given by

$$\mathcal{S}(t) := \left[\mathcal{W}_{1,0}(t_1, x)\right] \mathcal{S} \left[\mathcal{W}_{2,0}(t_2, x)\right]^{-1}$$

With

$$\mathcal{W}_{1,0}(t_1, x) = \left[\exp\left(\sum_{j=0}^{\infty} t_{1,j}\,\mathfrak{X}^j\right)\right] \quad \mathcal{W}_{2,0}(t_2, x) = \left[\exp\left(-\sum_{j=0}^{\infty} t_{2,j}\,\left(\mathfrak{X}^\top\right)^j\right)\right]$$

$$\exp(t\mathfrak{X}) = \begin{pmatrix} \binom{0}{0}t^0 & \binom{1}{0}t^1 & \binom{2}{0}t^2 & \binom{3}{0}t^3 & \dots \\ 0 & \binom{1}{1}t^{1-1} & \binom{2}{1}t^{2-1} & \binom{3}{1}t^{3-1} & \dots \\ 0 & 0 & \binom{2}{2}t^{2-2} & \binom{3}{2}t^{3-2} & \dots \\ 0 & 0 & 0 & \binom{3}{3}t^{3-3} & \dots \\ & & & & \ddots \end{pmatrix} \exp(tx)$$

Sobolev biorthogonality and integrable systems

We assume the Gauss–Borel factorization holds

$$G^t = (S_1^t)^{-1} H^t (S_2^t)^{-\top}$$

Deformed Sobolev biorthogonality

The time-dependent matrix polynomials

$$P_1^t(x) = S_1^t \chi(x), \qquad P_2^t(y) = S_2^t \chi(y)$$

are biorthogonal

$$\big(P_{1,n}^t(x), P_{2,m}^t(y)\big)_{\mathcal{S}^t} = \delta_{n,m} H_n^t$$

Deformed second kind functions

The t.dependent second kind functions are

$$C^t_{1,n}(z) = \left(P^t_{1,n}(x), \frac{1}{z-y} \right)_{g^t} \quad \left(C^t_{2,n}(z)\right)^\top = \left(\frac{1}{z-x}, P^t_{2,n}(y) \right) u^t$$

Deformed Christoffel–Darboux kernels

The t-dependent Christoffel–Darboux kernel and its mixed versions are

$$K^t_n(x,y) = \sum_{k=0}^{n} (P^t_{2,k}(y))^\top (H^t_k)^{-1} P^t_{1,k}(x)$$

Deformed second kind functions

The t.dependent second kind functions are

$$C^t_{1,n}(z)=\left(P^t_{1,n}(x),\frac{1}{z-y}\right)_{\mathcal{S}^t}\quad\left(C^t_{2,n}(z)\right)^\top=\left(\frac{1}{z-x},P^t_{2,n}(y)\right)u^t$$

Deformed Christoffel–Darboux kernels

The t-dependent Christoffel–Darboux kernel and its mixed versions are

$$K^t_n(x,y)=\sum_{k=0}^{n}(P^t_{2,k}(y))^\top(H^t_k)^{-1}P^t_{1,k}(x)$$

Sobolev biorthogonality and integrable systems

Sato–Wilson equations

$$
\begin{aligned}
\frac{\partial S_1}{\partial t_{1,j}}(S_1)^{-1} &= -\Big(S_1 \Lambda^j (S_1)^{-1}\Big)_- \\
\frac{\partial S_1}{\partial t_{2,j}}(S_1)^{-1} &= \Big(\tilde{S}_2 (\Lambda^\top)^j (\tilde{S}_2)^{-1}\Big)_- \\
\frac{\partial \tilde{S}_2}{\partial t_{1,j}}(\tilde{S}_2)^{-1} &= \Big(S_1 \Lambda^j (S_1)^{-1}\Big)_+ \\
\frac{\partial \tilde{S}_2}{\partial t_{2,j}}(\tilde{S}_2)^{-1} &= -\Big(\tilde{S}_2 (\Lambda^\top)^j (\tilde{S}_2)^{-1}\Big)_+
\end{aligned}
$$

with $\tilde{S}_2 = S_2 H$

Here $(A)_-$ is the projection of the matrix A onto the space of strictly lower triangular matrices while $(A)_+$ is its projection onto the space of upper triangular matrices

Proof:

$$
\begin{aligned}
-(S_1^t)^{-1}\frac{\partial S_1^t}{\partial t_{1,j}}(S_1^t)^{-1}\tilde{S}_2^t+(S_1^t)^{-1}\frac{\partial \tilde{S}_2^t}{\partial t_{1,j}} &= \Lambda^j G^t\\
&= \Lambda^j (S_1^t)^{-1}\tilde{S}_2^t,\\
-(S_1^t)^{-1}\frac{\partial S_1^t}{\partial t_{2,j}}(S_1^t)^{-1}\tilde{S}_2^t+(S_1^t)^{-1}\frac{\partial \tilde{S}_2^t}{\partial t_{2,j}} &= -G^t(\Lambda^j)^\top\\
&= (S_1^t)^{-1}\tilde{S}_2^t(\Lambda^j)^\top
\end{aligned}
$$

so that

$$
\begin{aligned}
-\frac{\partial S_1^t}{\partial t_{1,j}}(S_1^t)^{-1}+\frac{\partial \tilde{S}_2^t}{\partial t_{1,j}}\big(\tilde{S}_2^t\big)^{-1} &= S_1^t\Lambda^j(S_1^t)^{-1},\\
-\frac{\partial S_1^t}{\partial t_{2,j}}(S_1^t)^{-1}+\frac{\partial \tilde{S}_2^t}{\partial t_{2,j}}\big(\tilde{S}_2^t\big)^{-1} &= -\tilde{S}_2^t(\Lambda^j)^\top\big(\tilde{S}_2^t\big)^{-1}
\end{aligned}
$$

Proof:

$$
\begin{aligned}
-(S_1^t)^{-1}\frac{\partial S_1^t}{\partial t_{1,j}}(S_1^t)^{-1}\tilde{S}_2^t+(S_1^t)^{-1}\frac{\partial \tilde{S}_2^t}{\partial t_{1,j}} &= \Lambda^j G^t \\
&= \Lambda^j (S_1^t)^{-1}\tilde{S}_2^t, \\
-(S_1^t)^{-1}\frac{\partial S_1^t}{\partial t_{2,j}}(S_1^t)^{-1}\tilde{S}_2^t+(S_1^t)^{-1}\frac{\partial \tilde{S}_2^t}{\partial t_{2,j}} &= -G^t (\Lambda^j)^\top \\
&= (S_1^t)^{-1}\tilde{S}_2^t(\Lambda^j)^\top
\end{aligned}
$$

so that

$$
\begin{aligned}
-\frac{\partial S_1^t}{\partial t_{1,j}}(S_1^t)^{-1}+\frac{\partial \tilde{S}_2^t}{\partial t_{1,j}}\big(\tilde{S}_2^t\big)^{-1} &= S_1^t\Lambda^j(S_1^t)^{-1}, \\
-\frac{\partial S_1^t}{\partial t_{2,j}}(S_1^t)^{-1}+\frac{\partial \tilde{S}_2^t}{\partial t_{2,j}}\big(\tilde{S}_2^t\big)^{-1} &= -\tilde{S}_2^t(\Lambda^j)^\top\big(\tilde{S}_2^t\big)^{-1}
\end{aligned}
$$

2D Toda lattice equations

Using $t_{1,1} = \eta$ and $t_{2,1} = \zeta$,

$$\frac{\partial}{\partial \zeta}\Big(\frac{\partial h_k}{\partial \eta}(h_k)^{-1}\Big) + h_{k+1}(h_k)^{-1} - h_k(h_{k-1})^{-1} = 0,$$

Symmetric case: reduction the non-Abelian 1D Toda lattice equation, where $\eta = \zeta$,

$$\frac{\partial}{\partial \eta}\Big(\frac{\partial h_k}{\partial \eta}(h_k)^{-1}\Big) + h_{k+1}(h_k)^{-1} - h_k(h_{k-1})^{-1} = 0.$$

Proof:

$$\frac{\partial h_k}{\partial \eta}(H_k)^{-1} = U_k - U_{k+1}, \qquad k \in \{0, 1, \dots\},$$

$$\frac{\partial U_k}{\partial \zeta} = h_k(h_{k-1})^{-1}, \qquad k \in \{1, 2, \dots\}$$

where U_k, $k = 1, 2, \dots$, are $U_0 = 0$ and $U_k := (S_1^t)_{k,k-1}$, $k \in \{1, 2, \dots\}$

If $h_k = \mathrm{e}^{\varphi_k}$

$$\frac{\partial^2 \varphi_k}{\partial \xi \partial \eta} + \mathrm{e}^{\varphi_{k+1}-\varphi_k} - \mathrm{e}^{\varphi_k - \varphi_{k-1}} = 0.$$

Symmetric case:

$$\frac{\partial^2 \varphi_k}{\partial \eta^2} + \mathrm{e}^{\varphi_{k+1}-\varphi_k} - \mathrm{e}^{\varphi_k - \varphi_{k-1}} = 0.$$

They mix nearest neighbors $k-1, k, k+1$ in the 1D lattice

Sobolev biorthogonality and integrable systems

If $h_k = \mathrm{e}^{\varphi_k}$

$$\frac{\partial^2 \varphi_k}{\partial \xi \partial \eta} + \mathrm{e}^{\varphi_{k+1} - \varphi_k} - \mathrm{e}^{\varphi_k - \varphi_{k-1}} = 0.$$

Symmetric case:

$$\frac{\partial^2 \varphi_k}{\partial \eta^2} + \mathrm{e}^{\varphi_{k+1} - \varphi_k} - \mathrm{e}^{\varphi_k - \varphi_{k-1}} = 0.$$

They mix nearest neighbors $k-1, k, k+1$ in the 1D lattice

Lax and Zakharov–Shabat matrices

$$L_1 := S_1 \Lambda (S_1)^{-1}, \qquad L_2 := \tilde{S}_2 (\Lambda)^\top (\tilde{S}_2)^{-1}$$

$$B_{1,j} := \big((L_1)^j\big)_+, \qquad B_{2,j} := \big((L_2)^j\big)_-$$

The zero-curvature formulation of the integrable hierarchy

The Lax matrices are subject to the following

$$\frac{\partial L_i}{\partial t_{j,k}} = \big[B_{j,k}, L_i\big]$$

and Zakharov–Sabat matrices fulfill the following *Zakharov–Shabat equations*

$$\frac{\partial B_{i',k'}}{\partial t_{i,k}} - \frac{\partial B_{i,k}}{\partial t_{i',k'}} + \big[B_{i,k}, B_{i',k'}\big] = 0$$

Sobolev biorthogonality and integrable systems

Lax and Zakharov–Shabat matrices

$$L_1 := S_1 \Lambda (S_1)^{-1}, \qquad L_2 := \tilde{S}_2 (\Lambda)^\top (\tilde{S}_2)^{-1}$$

$$B_{1,j} := \big((L_1)^j\big)_+, \qquad B_{2,j} := \big((L_2)^j\big)_-$$

The zero-curvature formulation of the integrable hierarchy

The Lax matrices are subject to the following

$$\frac{\partial L_i}{\partial t_{j,k}} = \big[B_{j,k}, L_i\big]$$

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In contrast with what happens in the standard theory of deformation of moment matrices, where $L_1 = L_2$ (because both coincide with the tri-diagonal Jacobi matrix responsible for the usual three term recurrence relation), this is no longer the case in the Sobolev context.

$\Lambda G_{\mathcal{S}} \neq G_{\mathcal{S}} \Lambda^{\top}$ and $L_1 \neq L_2$ and we can only infer that L_1 and L_2 are Hessenberg matrices

Wave matrices

$$W_1^t := S_1^t W_{1,0}^{t_1}, \qquad \tilde{W}_2^t := \left(\tilde{S}_2^t\right)^{-\top} W_{2,0}^{-t_2} = (H^t)^{-\top} S_2^t W_{1,0}^{t_2} \tag{2}$$

where $\tilde{S}_2^t := H^t \left(S_2^t\right)^{-\top}$.

Zakharov–Shabat equations

The wave matrices satisfy the linear systems

$$\frac{\partial W_1^t}{\partial t_{1,j}} = B_{1,j} W_1^t, \qquad \frac{\partial W_1^t}{\partial t_{2,j}} = B_{2,j} W_1^t$$

$$\frac{\partial \tilde{W}_2^t}{\partial t_{1,j}} = -(B_{1,j})^\top \tilde{W}_2^t \qquad \frac{\partial \tilde{W}_2^t}{\partial t_{2,j}} = -(B_{2,j})^\top \tilde{W}_2^t$$

Baker functions

$$
\begin{aligned}
\Psi_1(t,z) &:= W_1^t \chi(z) & \Psi_2^*(t,z) &:= \tilde{W}_2^t \chi(z) \\
\big(\Psi_1^*(t,z)\big)^\top &:= \big(\chi^*(z)\big)^\top G (\tilde{W}_2^t)^\top & \Psi_2(t,z) &:= W_1^t G \chi^*(z),
\end{aligned}
$$

Baker functions and the biorthogonal polynomials

$$\Psi_1(t,z) = \mathrm{e}^{t_1(z)}\, P_1^t(z)$$

$$\Psi_2^*(t,z) := \mathrm{e}^{-t_2(z)}(H^t)^{-\top} P_2^t(z)$$

$$\big(\Psi_1^*(t,z)\big)^\top := \left(\frac{1}{z-x}, \mathrm{e}^{-t_2(y)}\, P_2^t(y)\right)_{\mathcal{S}} (H^t)^{-1}$$

$$\Psi_2(t,z) = \left(\mathrm{e}^{t_1(x)}\, P_1^t(x), \frac{1}{z-y}\right)_{\mathcal{S}}$$

Zakharov–Shabat equations

The Baker functions satisfy the linear systems

$$
\begin{aligned}
\frac{\partial \Psi_1}{\partial t_{1,j}} &= B_{1,j}\Psi_1, & \frac{\partial \Psi_1}{\partial t_{2,j}} &= B_{2,j}\Psi_1 \\
\frac{\partial \Psi_2^*}{\partial t_{1,j}} &= -\left(B_{1,j}\right)^\top \Psi_2^*, & \frac{\partial \Psi_2^*}{\partial t_{2,j}} &= -\left(B_{2,j}\right)^\top \Psi_2^* \\
\frac{\partial \left(\Psi_1^*\right)^\top}{\partial t_{1,j}} &= -\left(\Psi_1^*\right)^\top B_{1,j}, & \frac{\partial \left(\Psi_1^*\right)^\top}{\partial t_{2,j}} &= -\left(\Psi_1^*\right)^\top B_{2,j} \\
\frac{\partial \Psi_2}{\partial t_{1,j}} &= \left(B_{1,j}\right)^\top \Psi_2, & \frac{\partial \Psi_2}{\partial t_{2,j}} &= \left(B_{2,j}\right)^\top \Psi_2
\end{aligned}
$$

Asymptotic module. I

Given two semi-infinite matrices $Z_1(t)$ and $Z_2(t)$ we say that

- $Z_1(t) \in \mathfrak{l} W_{0,1}^{t_1}$ if $Z_1(t)\big(W_{0,1}^{t_1}\big)^{-1}$ is a block strictly lower triangular matrix.
- $Z_2(t) \in \mathfrak{u}$ if $Z_2(t)$ is a block upper triangular matrix.

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Asymptotic module. II

Given two semi-infinite matrices $Z_1(t)$ and $Z_2(t)$ such that

- $Z_1(t) \in \mathfrak{l} W_{1,0}^{t_1}$,
- $Z_2(t) \in \mathfrak{u}$,
- $Z_1(t) G = Z_2(t)$.

then $Z_1(t) = Z_2(t) = 0$

Proof: Observe that

$$Z_1(t)\big(W_{0,1}^{t_1}\big)^{-1}\big(S_1(t)\big)^{-1} = Z_2(t)\big(\tilde{S}_2(t)\big)^{-1},$$

and, as in the LHS we have a strictly lower triangular block semi-infinite matrix while in the RHS we have an upper triangular block semi-infinite matrix, both sides must vanish and the result follows

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Notation

- When $A - B \in \mathfrak{l}W_{0,1}^{t_1}$ we write $A = B + \mathfrak{l}W_{0,1}^{t_1}$ and if $A - B \in \mathfrak{u}$ we write $A = B + \mathfrak{u}$
- We put all the times $t_{2,j} = 0$ and consider only continuous deformation given by the times $t_{1,j}$, $j \in \{1, 2, \dots\}$, and our first three times will be denoted by $\eta := t_{1,1}$, $\rho := t_{1,2}$ and $\theta := t_{1,3}$, $U_k := (S_1)_{k,k-1}$, $k \in \{1, 2 \dots\}$

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Second and third order linear ODE

Among others the Baker function Ψ_1 satisfies the following linear differential equations

$$\frac{\partial(\Psi_1)_k}{\partial\rho} = \frac{\partial^2(\Psi_1)_k}{\partial\eta^2} - 2\frac{\partial U_k}{\partial\eta}(\Psi_1)_k$$

$$\frac{\partial(\Psi_1)_k}{\partial\theta} = \frac{\partial^3(\Psi_1)_k}{\partial\eta^3} - 3\frac{\partial U_k}{\partial\eta}\frac{\partial(\Psi_1)_k}{\partial\eta} - \frac{3}{2}\Big(\frac{\partial^2 U_k}{\partial\eta^2} + \frac{\partial U_k}{\partial\rho}\Big)(\Psi_1)_k$$

Proof:

$$\frac{\partial W_1}{\partial \rho} = \Big(\frac{\partial S_1}{\partial \rho} + S_1 \Lambda^2\Big) W_{\mathbf{0},\mathbf{1}}^{t_1}$$

$$\frac{\partial^2 W_1}{\partial \eta^2} = \Big(\frac{\partial^2 S_1}{\partial \eta^2} + 2\frac{\partial S_1}{\partial \eta}\Lambda + S_1 \Lambda^2\Big) W_{\mathbf{0},\mathbf{1}}^{t_1}$$

so that

$$\left(\frac{\partial}{\partial \rho} - \frac{\partial^2}{\partial \eta^2}\right)(W_1) = -2\left(\frac{\partial U}{\partial \eta}\Lambda\right) W_{\mathbf{0},\mathbf{1}}^{t_1} + \mathfrak{l} W_{\mathbf{0},\mathbf{1}}^{t_1}$$

and, consequently,

$$Z_1 := \left(\frac{\partial}{\partial \rho} - \frac{\partial^2}{\partial \eta^2} + 2\frac{\partial U}{\partial \eta}\Lambda\right)(W_1) \in \mathfrak{l} W_{\mathbf{0},\mathbf{1}}^{t_1}$$

On the other hand,

$$Z_2 := \frac{\partial \tilde{S}_2}{\partial \rho} - \frac{\partial^2 \tilde{S}_2}{\partial \eta^2} + 2\frac{\partial U}{\partial \eta}\Lambda \tilde{S}_2 \in \mathfrak{u}$$

KP equation

The compatibility of both equations leads to

$$\frac{\partial}{\partial \eta}\Big(4\frac{\partial U_k}{\partial \theta} + 6\Big(\frac{\partial U_k}{\partial \eta}\Big)^2 - \frac{\partial U_k}{\partial \eta^3}\Big) - \frac{\partial^2 U_k}{\partial \rho^2} = 0$$

1. Is a non linear equation for the first nontrivial coefficients of the monic orthogonal polynomials $P_{1,k}(x) = x^k + U_k x^{k-1} + \cdots$
2. The equation involves only one site, not nearest neighbors as in the Toda lattice

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