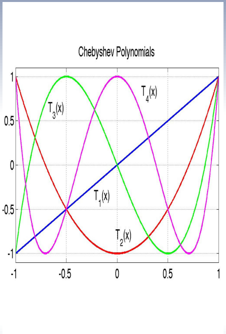

Chebyshev Polynomials, I

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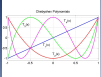
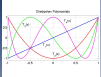
III Escuela Orthonet – Nov 19, 2018



Pafnuty Lvovich Chebyshev

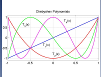
1821–1894



Introduction

From among the many research subjects that I encountered in studying and comparing different mechanisms of motion transfer, especially in a ***steam engine****, where efficiency and reliability depend much on the way the power of steam is transferred, I was especially occupied by the theory of mechanisms known as parallelograms.....*
While trying to derive the rules for constructing specific parallelograms directly from their properties, I encountered ***problems in analysis*** *that were not well known then.*

— P. L. Chebyshev

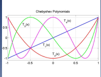


Introduction

As formulated in "Théorie des mécanismes connus sous le nom de parallélogrammes (1854)", the general problem is:

> *Given a continuous function f, find a polynomial of a given degree n such that the maximum of its deviation from $f(x)$ in a given interval is smaller than that of all other polynomials of the same degree.*

In other words, given an interval $[a, b]$, one has to find the coefficients α_i of $P(x) = \alpha_n x^n + \ldots + \alpha_1 x + \alpha_0$ so that the expression $\max_{a \leq x \leq b} |f(x) - P(x)|$ is minimized.



Introduction

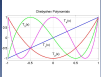
But how do we find the approximating polynomial $P(x)$?

The difference $f(x) - P(x)$ is, according to Chebyshev, known to necessarily have the following property:

> *The set of its absolute values of maxima and minima in the given interval contains the same number at least $n+2$ times.*

He never gives a proof of this statement, nor does he discuss existence and uniqueness, but he sets up a system of algebraic equations from which one can determine $P(x)$.

Certain particular cases, e.g., $f(x) = x^{n+p}$ with $p \geq 1$, can also be solved directly.



Our setting

In this course, we shall merely focus on how to approximate x^n by polynomials of lower degree.

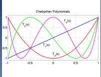
But we shall do so on fairly general sets, not just intervals!

Let $\mathsf{E} \subset \mathbb{R}$ (or $\mathbb{C}$) be an infinite, compact set of points. We sometimes assume that $\mathrm{Cap}(\mathsf{E}) > 0$ and suppose that E is *regular* (for potential theory).

The **Chebyshev polynomial of degree** n is the monic polynomial T_n with

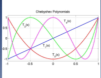
$$t_n := \|T_n\|_{\mathsf{E}} = \inf\Big\{ \|P\|_{\mathsf{E}} \,:\, \deg(P) = n \text{ and } P \text{ is monic} \Big\}.$$

Here, $\|\cdot\|_{\mathsf{E}}$ denotes the sup-norm on E.

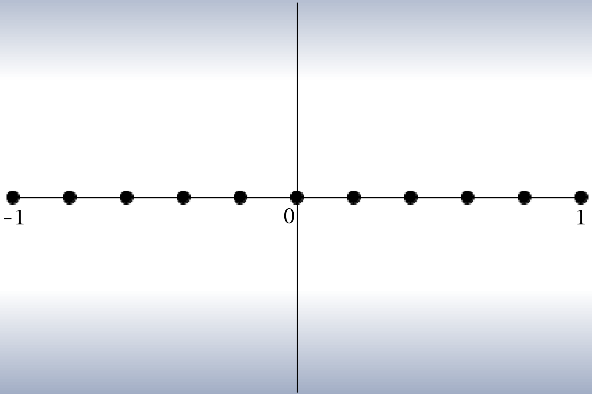


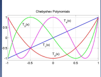
Timeline

- 1854: Chebyshev – one interval (monic polynomials)
- 1868: Zolotarev – fix 2 coefficients $(z^n + \sigma z^{n-1} + \ldots)$
- 1919: Faber – analytic Jordan regions (strong asymptotics)
- 1924: Fekete, Szegő – compact subsets of $\mathbb{C}$ (weak asymp.)
- 1928: Akhiezer – fix 3 coefficients $(z^n + \sigma z^{n-1} + \tau z^{n-2} + \ldots)$
- 1931: Akhiezer – two intervals (periodic/almost periodic)
- 1960: Meyman – fix *any* number of coefficients
- 1969: Widom – finitely many smooth regions (strong asymp.)

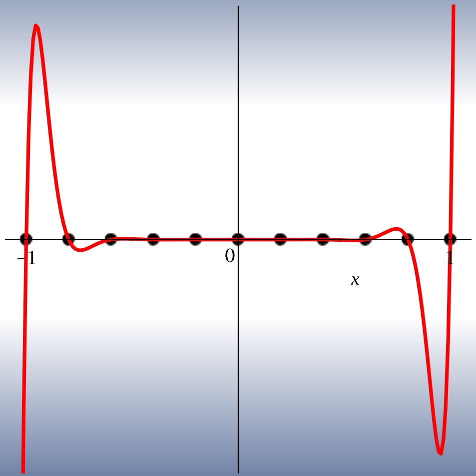


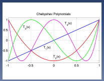
Example: $E = [-1, 1]$



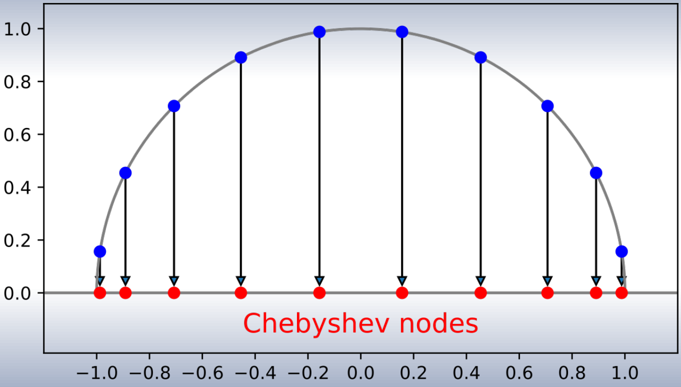


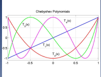
Example, cont.





Example, cont.





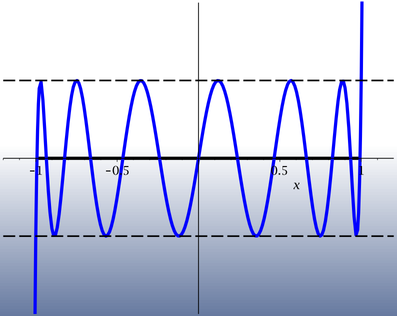
Example, cont.

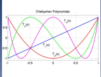
When E $= [-1, 1]$, the case considered by Chebyshev, the polynomials in question are

$$T_0(x) = 1, \quad T_n(x) = 2^{-n+1}\cos(n\theta) \text{ for } n \geq 1,$$

with $x = \cos(\theta)$.

The graph of T_{13}:



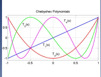


The French connection

I assume that I am not the only one, who does not understand the interest in and significance of these strange problems on maxima and minima studied by Chebyshev in memoirs whose titles often begin with "On functions deviating least from zero...". Could it be that one must have a slavic soul to understand the great Russian Scholar?

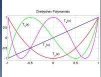
— H. Lebesgue

This quote is a little bizarre given that, as we'll see, E. Borel (Lebesgue's thesis advisor) made important contributions to the subject in 1905!



YouTube

See `https://m.youtube.com/watch?v=bU8rDuX4GBU`



The alternation theorem

We say that P_n, a real degree n polynomial, has an **alternating set** in $\mathsf{E} \subset \mathbb{R}$ if there exists $\{x_j\}_{j=0}^{n} \subset \mathsf{E}$ with

$$x_0 < x_1 < \ldots < x_n$$

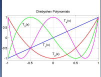
so that

$$P_n(x_j) = (-1)^{n-j} \|P_n\|_{\mathsf{E}}, \quad j = 0, \ldots, n.$$

Theorem (Borel and Markov, independently, 1905)

A Chebyshev polynomial for E *has an alternating set in* E*.*

Conversely, if a monic degree n polynomial has an alternating set in E*, then it is the nth Chebyshev polynomial for* E*.*



Consequences of AT

The alternation theorem implies uniqueness of the Chebyshev polynomials.

- Suppose P_n and Q_n are two distinct minimizers and let

  $$T_n = \frac{1}{2}(P_n + Q_n).$$

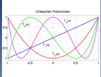
  Since $\|T_n\|_{\mathsf{E}} \leq \max\{\|P_n\|_{\mathsf{E}}, \|Q_n\|_{\mathsf{E}}\}$, T_n is also a minimizer.

  So T_n has an alternating set in E and we can hence pick $x_0 < x_1 < \ldots < x_n$ in E such that

  $$|T_n(x_j)| = \|T_n\|_{\mathsf{E}} \ \text{ for } \ j = 0, 1, \ldots, n.$$

  Since $|P_n(x_j)|, |Q_n(x_j)| \leq \|T_n\|_{\mathsf{E}}$ and $\frac{1}{2}|P_n(x_j) + Q_n(x_j)| = |T_n(x_j)| = \|T_n\|_{\mathsf{E}}$, we have $Q_n(x_j) = P_n(x_j) = T_n(x_j)$ for all j.

  Thus, $P_n - Q_n$ has at least $n + 1$ zeros and hence $P_n = Q_n$.



Consequences of AT

The theorem also implies several facts about the zeros of T_n.

For if $x_0 < x_1 < \ldots < x_n$ is an alternating set for T_n, then because of the sign change there must be at least one zero (in $\mathbb{R}$, not necessarily in E) between x_{j-1} and x_j for $j = 1, \ldots, n$.

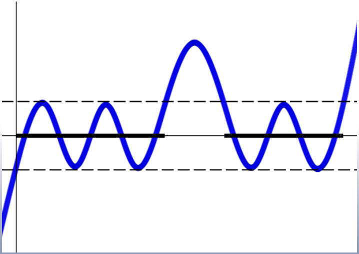
As this accounts for all n zeros of T_n, we conclude:

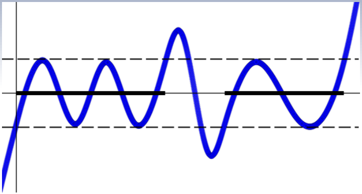
- All the zeros of T_n are real and simple and they lie in cvh(E), the convex hull of E.
- Each gap of E (i.e., a bounded connected component of $\mathbb{R} \setminus \mathrm{E}$) has at most one zero of T_n.
- At the endpoints of cvh(E), we have that $|T_n(x)| = \|T_n\|_{\mathrm{E}}$.

Two intervals

It may happen that T_n has no zeros in a gap of E.

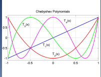
$\downarrow$





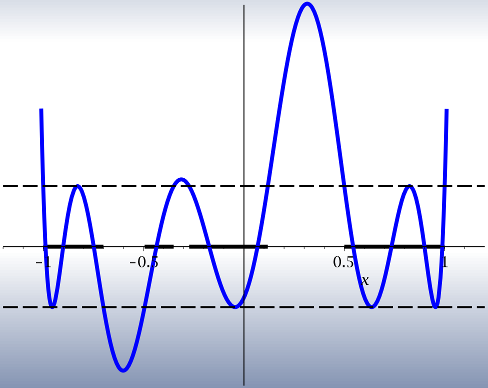
$\uparrow$

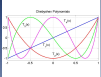
But due to alternation, it never has more than one zero in any gap of E.



Several intervals

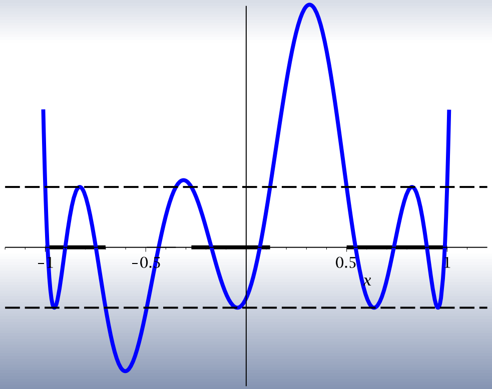
The graph of T_{10} for E a union of 4 intervals.

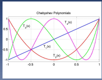




Several intervals

The graph of T_{10} for E a union of 4 intervals — or only 3?





Period-n sets

Given an infinite compact set $\mathsf{E} \subset \mathbb{R}$, we define

$$\mathsf{E}_n := T_n^{-1}\big([-t_n, t_n]\big), \quad \mathrm{int}(\mathsf{E}_n) := T_n^{-1}\big((-t_n, t_n)\big).$$

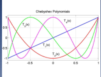
and note that we always have $\mathsf{E} \subset \mathsf{E}_n \subset \mathrm{cvh}(\mathsf{E})$. In fact:

There exist $\alpha_1 < \beta_1 \leq \alpha_2 < \cdots \leq \alpha_n < \beta_n$ so that

$$\mathrm{int}(\mathsf{E}_n) = \bigcup_{j=1}^{n} (\alpha_j, \beta_j), \quad \mathsf{E}_n = \bigcup_{j=1}^{n} [\alpha_j, \beta_j].$$

Moreover,

- α_1 and β_n belong to E,
- on (α_j, β_j), we have that $(-1)^{n-j} T_n'(x) > 0$,
- for each $j \leq n-1$, at least one of β_j and α_{j+1} lie in E.



Potential Theory 101

Chebyshev polynomials are intimately connected with 2-dimensional potential theory. So let's review some of the basics of that subject.

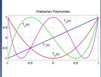
Given a probability measure $d\mu$ of compact support in $\mathbb{C}$, we define its logarithmic **potential** by

$$\Phi_\mu(z) = \int \log|z-w|^{-1} d\mu(w)$$

and its potential (or Coulomb) **energy** by

$$\mathcal{E}(\mu) = \int \Phi_\mu(z) d\mu(z) = \iint \log|z-w|^{-1} d\mu(w) d\mu(z).$$

Note that $\mathcal{E}(\mu)$ is either finite or diverges to $+\infty$.



Capacity

We define the **Robin constant** of a compact set $E \subset \mathbb{C}$ by

$$R(E) = \inf\{\mathcal{E}(\mu) \mid \text{supp}(\mu) \subset E \text{ and } \mu(E) = 1\}.$$

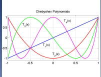
If $R(E) = \infty$, we say E is a **polar set** or has capacity zero.

If something holds except for a polar set, we say it holds q.e. (for quasi-everywhere).

Finally, we define the logarithmic **capacity** of E by

$$\text{Cap}(E) = \exp\Big\{-R(E)\Big\}.$$

This is of course that same as $R(E) = -\log\Big(\text{Cap}(E)\Big)$.



Equilibrium

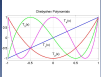
If $\mathrm{E} \subset \mathbb{C}$ is non-polar, then one can show there is a unique probability measure $d\rho_{\mathrm{E}}$ with $\mathrm{supp}(\rho_{\mathrm{E}}) \subset \mathrm{E}$ whose potential energy is $R(\mathrm{E})$.

This is the so-called **equilibrium measure** of E.

Existence follows from weak lower semicontinuity of $\mathcal{E}(\cdot)$ and weak compactness of the family of probability measures.

Uniqueness is then a consequence of the fact that $\mathcal{E}(\cdot)$ is strictly convex on the set of probability measures.

The function $\Phi_{\mathrm{E}}(z) = \int \log|z - w|^{-1} d\rho_{\mathrm{E}}(w)$ is called the **equilibrium potential** and will appear again shortly.



Harmonic measure

We also refer to $d\rho_{\mathsf{E}}$ as **harmonic measure** of E (from ∞).

Since $\mathsf{E} \subset \mathbb{C}$ is non-polar, there is a unique solution to the **Dirichlet problem** for $\overline{\mathbb{C}} \setminus \mathsf{E}$. That is,

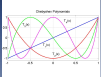
- for any continuous function $f : \mathsf{E} \to \mathbb{R}$, there exists a unique function u_f which is harmonic on $\overline{\mathbb{C}} \setminus \mathsf{E}$ and which approaches $f(\zeta)$ for q.e. $\zeta \in \mathsf{E}$.

The point is now that, in fact, $u_f(\infty) = \int_{\mathsf{E}} f(\zeta) d\rho_{\mathsf{E}}(\zeta)$.

More generally,

$$u_f(z) = \int_{\mathsf{E}} f(\zeta) d\rho_{\mathsf{E}}(z, \zeta) \ \text{ for all } z \in \overline{\mathbb{C}} \setminus \mathsf{E},$$

where $d\rho_{\mathsf{E}}(z, \cdot)$ is harmonic measure (from z).



Green's function

The **Green's function** of a non-polar compact set $\mathrm{E} \subset \mathbb{C}$ is defined by

$$g_{\mathrm{E}}(z) = R(\mathrm{E}) - \Phi_{\mathrm{E}}(z),$$

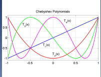
where Φ_{E} is the equilibrium potential (defined previously).

This is the unique function harmonic on $\mathbb{C} \setminus \mathrm{E}$ with q.e. boundary value 0 on E and so that $g_{\mathrm{E}}(z) - \log|z|$ is harmonic at ∞.

Moreover, $g_{\mathrm{E}}(z) \geq 0$ everywhere and near ∞ we have

$$g_{\mathrm{E}}(z) = \log|z| + R(\mathrm{E}) + \mathcal{O}(1/|z|).$$

If g_{E} is zero on E and continuous on all of $\mathbb{C}$, we say that the set E is **regular** (for potential theory).



Period-n sets, cont.

Let us define

$$\Delta_n(z) := 2T_n(z)/\|T_n\|_{\mathsf{E}}$$

so that E_n is exactly the set where $-2 \le \Delta_n(x) \le 2$ and Δ_n takes values in $\mathbb{C} \setminus [-2, 2]$ on $\mathbb{C} \setminus \mathsf{E}_n$.

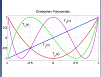
Theorem

The Green's function of E_n is given by

$$g_n(z) = \frac{1}{n} \log \left| \frac{\Delta_n(z)}{2} + \sqrt{\left(\frac{\Delta_n(z)}{2}\right)^2 - 1} \right|.$$

For z near ∞, the argument inside log behaves like $2z^n/\|T_n\|_{\mathsf{E}}$.

It thus follows that $t_n := \|T_n\|_{\mathsf{E}} = 2\,\mathrm{Cap}(\mathsf{E}_n)^n$.



Equilibrium measure of E_n

Since Δ_n runs monotonically from -2 to 2 (or vice versa) on every band of E_n, we have the following:

Theorem

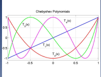
In each band of E_n, define $\theta(x) \in [0, \pi]$ by

$$\Delta_n(x) = 2\cos(\theta(x)).$$

Then the equilibrium measure of E_n is given by

$$d\rho_n(x) = (\pi n)^{-1} |\theta'(x)| dx.$$

In particular, each band has ρ_n-measure $1/n$. Moreover, if η_j is the zero of T_n in $[\alpha_j, \beta_j]$, then both $[\alpha_j, \eta_j]$ and $[\eta_j, \beta_j]$ have ρ_n-measure $1/2n$.



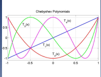
Equilibrium measure, cont.

It follows from the explicit formula for the Green's function that

$$-\int \frac{d\rho_n(x)}{x-z} = \frac{1}{n}\frac{\Delta_n'(z)}{\sqrt{\Delta_n(z)^2-4}}.$$

One can use this to deduce, via the Stieltjes–Perron inversion formula, that $d\rho_n$ is absolutely continuous on E_n with respect to dx and given by

$$\frac{d\rho_n(x)}{dx} = \frac{1}{\pi n}\frac{|\Delta_n'(x)|}{\sqrt{4-\Delta_n(x)^2}}, \quad x \in \mathrm{E}_n.$$



Gaps of E

Though it looks innocent, the following result is very useful.

Theorem

Let K be a gap of E (i.e., a bounded connected component of $\mathbb{R} \setminus \mathrm{E}$). Then $\rho_n(K) \leq 1/n$.

If T_n has no zero in K, then $1/n$ can be replaced by $1/2n$.

Moreover, $K \cap \mathrm{E}_n$ is either empty or a single interval.

To see this, recall that K contains at most one band of E_n. And if T_n has no zero in K, at most half a band lies in K.

If $K \cap \mathrm{E}_n \neq \emptyset$, it can either be a closed, half open, or open interval.