Congreso de la Real Sociedad Matemática Española
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On the computation of orthogonal rational functions

Bernhard Beckermann[1], **Karl Deckers**[1] & **Miroslav Pranić**[2]

Several techniques are known to compute a new orthogonal polynomial φ_{k+1} of degree $k+1$ from $\mathcal{L}_k := \text{span}\{\varphi_0, ..., \varphi_k\}$ in case of (discrete) orthogonality on the real line. In the Arnoldi approach one chooses $\Phi_k \in \mathcal{L}_k$ and makes $x\Phi_k$ orthogonal against $\varphi_0, ..., \varphi_k$. By taking as Φ_k a linear combination of φ_k and the kernel (or GMRES) polynomial $\psi_k(x) = \sum_{j=0}^{k} \varphi_j(0)\varphi_j(x)$, one needs to orthogonalize only against $\varphi_{k-2}, \varphi_{k-1}, \varphi_k$, and obtains what in numerical linear algebra is called Orthores, Orthomin or SymLQ [1]. A construction of an orthogonal basis of rational Krylov subspaces for given prescribed poles z_j can be done via orthogonal rational functions (ORF) [2], and is required for instance in the approximate computation of matrix functions. Here, following [4], the choice of the continuation vector Φ_k which is multiplied by $x/(x - z_{k+1})$ becomes essential, for instance for preserving orthogonality in a numerical setting. By generalizing the techniques of [2, 3], we compare several approaches and find optimal ones.

Keywords: orthogonal rational functions, rational Arnoldi, continuation vector

Mathematics Subject Classification 2010: 65F25, 42C05

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On orthogonal polynomials with respect to a differential operator

Jorge Alberto Borrego Morell[1],

We consider orthogonal polynomials with respect to a linear differential operator

$$L^{(M)} = \sum_{k=0}^{M} \rho_k(z) \frac{d^k}{dz^k},$$

where $\{\rho_k\}_{k=0}^{M}$ are complex polynomials such that $deg[\rho_k] \leq k, 0 \leq k \leq M$, with equality for at least one index. We analyze the uniqueness and zero location of these polynomials. An interesting phenomena occurring in this kind of orthogonality is the existence of operators for which the associated sequence of orthogonal polynomials reduces to a finite set. For a given operator, we find a classification of the measures for which it is possible to guarantee the existence of an infinite sequence of orthogonal polynomials, in terms of a linear system of difference equations with varying coefficients. Also, for the case of a first order differential operator, we locate the zeros and establish the strong asymptotic behavior of these polynomials.

Keywords: Orthogonal polynomials, linear differential operators, zero location, asymptotic behavior.

Mathematics Subject Classification 2010: 42C05, 47E05

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Integral representations of some Hermite type matrix-valued kernels and non-commutative Painlevé equations

Mattia Cafasso[1], **Manuel Domínguez de la Iglesia***[2]

We study double integral representations of kernels associated with some examples of Hermite type matrix-valued orthogonal polynomials. We show that these kernels are related through the Its-Izergin-Korepin-Slavnov (IIKS) theory with a certain Riemann-Hilbert problem. After an appropriate transformation, we obtain a Lax pair whose compatibility conditions lead to a non-commutative version of the Painlevé IV nonlinear differential equation.

Keywords: Matrix-valued orthogonal polynomials, Non-commutative Painlevé equations

Mathematics Subject Classification 2010: 31B10, 34M55

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Exceptional orthogonal polynomials

David Gómez-Ullate

Exceptional orthogonal polynomials are complete sets of orthogonal polynomials which arise as solutions of a Sturm-Liouville problem and have gaps in their degree sequence. They extend in some sense the classical families of Hermite, Laguerre and Jacobi [1, 2]. The weight function is a classical weight divided by a polynomial with zeros outside the interval of orthogonality.

In particular, we will show how these families can be obtained from the classical ones by means of an algebraic Darboux transformation [3], a particular class of Darboux transformations that preserves the polynomial character of the eigenfunctions [4]. Exceptional orthogonal polynomials have zeros in the interval of orthogonality (regular zeros) plus some extra zeros outside this region (exceptional zeros). We will show some interlacing properties and asymptotic behaviour for both types of zeros [5]. Higher order or Darboux-Crum transformations can also be used to generate new families of exceptional orthogonal polynomials [6] and we will comment on a recently launched conjecture [7] that would pave the way towards a full classification of the whole class.

Keywords: orthogonal polynomials, differential equations, Darboux transformations

Mathematics Subject Classification 2010: Primary 33C45; Secondary 34B24, 42C05.

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Orthogonal polynomials in the normal matrix model

Arno Kuijlaars[1]

The normal matrix model is a random matrix model defined on complex matrices. The eigenvalues in this model fill a two-dimensional region in the complex plane as the size of the matrices tends to infinity. Orthogonal polynomials with respect to a planar measure are a main tool in the analysis.

In many interesting cases, however, the orthogonality is not well-defined, since the integrals that define the orthogonality are divergent. I will present a way to redefine the orthogonality in terms of a well-defined Hermitian form. This reformulation allows for a Riemann-Hilbert characterization as multiple orthogonal polynomials. For the special case of a cubic potential it is possible to do a complete steepest descent analysis on the Riemann-Hilbert problem, which leads to strong asymptotics of the multiple orthogonal polynomials, and in particular to the two-dimensional domain where the eigenvalues are supposed to accumulate.

This is joint work with Pavel Bleher (Indianapolis).

Keywords: Multiple orthogonal polynomials, Random matrices, Laplacian growth

Mathematics Subject Classification 2010: 42C05, 15B52, 31A35

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On two variable Koornwinder polynomials and three term relations

Misael E. Marriaga Castillo[1], **Teresa E. Pérez**[2], **Miguel A. Piñar**[2]

When polynomials in d variables are expressed in vector form ([1]), they satisfy exactly d three term relations with matrix coefficients. In this work we consider the Koorwinder's method ([3]) to construct orthogonal polynomials in two variables from orthogonal polynomials in one variable, and we study the two three term relations for these polynomials. We deduce the explicit expression for the matrix coefficients using the the three term recurrence relation for the involved univariate orthogonal polynomials. These matrices are diagonal or tridiagonal with entries computable from the relations in one variable.

Keywords: Orthogonal polynomials in two variables, three term relations

Mathematics Subject Classification 2010: 42C05, 33C50

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Matrix difference and q-difference operators having orthogonal polynomials as eigenfunctions.

Ana Martínez de los Ríos[1],

In the last decade a huge amount of examples of matrix orthogonal polynomials (MOP) which are eigenfunctions of second order differential operators with matrix polynomial coefficients has been constructed, [1]. In this talk we consider difference and q-difference operators:

$$D(P(x)) = P(x+1)F_1(x) + P(x)F_0(x) + P(x-1)F_{-1}(x),$$

$$D_q(P(x)) = P(qx)G_1(x) + P(x)G_0(x) + P(q^{-1}x)G_{-1}(x)$$

where F_{-1}, F_0 y F_1 are matrix polynomials in x and G_{-1}, G_0 y G_1 are matrix polynomials in x^{-1}, all of them of degree at most 2.

In the study of MOP being eigenvalues of such operators, the key concept is that of symmetry between an operator and a weight matrix W, [2]. We will stablish sufficient conditions that assure the symmetry of an operator D (respectively D_q) with respect to a weight matrix W. A method to construct weight matrices having difference operators (respectively q-difference) will be shown as well as some of the examples constructed with these methods.

Keywords: Matrix valued orthogonal polynomials, difference and q-difference operators.

Mathematics Subject Classification 2010: 47B39, 33C45, 42C05

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Convergence of tipe II Hermite-Padé approximants.

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Let $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ be a Nikishin system and Δ_1 be the convex hull of $supp(\sigma_1)$. Let $(r_1, \ldots, r_m)$ be rational functions such that $r_k(\infty) = 0$ and the poles of r_k lie in $\mathcal{C} \setminus \Delta_1$, for all $k = 1, \ldots, m$. We study the convergence of the diagonal sequence of type II Hermite-Padé approximants associated to the system of functions $(f_1, \ldots, f_m)$ where $f_k(z) = \int \frac{ds_k(x)}{z-x} + r_k$, $k = 1, \ldots, m$.

Keywords: Nikishin system, Type II Hermite-Padé approximants, Clave3

Mathematics Subject Classification 2010: 30E10,41A21,42C05

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Nuevos desarrollos en serie de las funciones hipergeometricas $_{p+1}F_p$

José L.López[1], **Pedro J. Pagola**[1]

Para evaluar las funciones hipergeométricas $_2F_1$ y $_3F_2$ podemos utilizar desarrollos en series de potencias. Estos desarrollos no son convergentes en todo el plano complejo. En el caso de la función hipergeométrica de Gauss, los puntos $e^{\pm i\pi/3}$ están excluidos siempre de los dominios de convergencia de los diferentes desarrollos conocidos. En el caso de la función $_3F_2$, el desarrollo en potencias que aparece en su definición unicamente converge en el disco unidad. En este trabajo hemos obtenido nuevos desarrollos de ambas funciones en serie de potencias convergentes en dominios mas amplios que los existentes hasta ahora. Además esta técnica es aplicable no solo a estas 2 funciones, sino a todas las funciones hipergeométricas generalizadas de la forma $_{p+1}F_p$.

Keywords: Gauss Hypergeometric Function, Hypergeometric Function $_3F_2$, Generalized Hypergeometric Functions$_{p+1}F_p$, Approximation by rational functions

Mathematics Subject Classification 2010: 33C05; 41A58; 41A20, 65D20

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Rational quadrature formulas on the interval and the unit circle

Francisco José Perdomo Pío[1],

Let μ be a measure on the interval $I = [-1, 1]$ and the integral

$$J_\mu(f) = \int_I f(x)d\mu(x), \tag{1}$$

where $J_\mu(f)$ will be estimated by means of quadrature formula on I,

$$J_n^\mu(f) = \sum_{j=1}^{n} \lambda_j f(x_j). \tag{2}$$

In the other hand, let $\mathring{\mu}$ be a measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and the integral

$$I_{\mathring{\mu}}(f) = \int_{\mathbb{T}} g(z)d\mathring{\mu}(z), \tag{3}$$

where $I_{\mathring{\mu}}(f)$ will be estimated by means of a quadrature formula on $\mathbb{T}$,

$$I_n^{\mathring{\mu}}(f) = \sum_{j=1}^{n} \mathring{\lambda}_j g(z_j). \tag{4}$$

When the functions f or g have polar singularities, is usual to choose the weights and the nodes in the quadrature formula, so that exact integrate in certain spaces of rational functions.

The aim of this talk, is to relate the integrals (1) and (3), and also the corresponding rational quadrature fórmulas (2) and (4). One or more nodes can be prefixed. In this way, we can enrichment the theory of Orthogonality and quadratures in both directions.

Keywords: Quadrature formulas, Orthogonality, Rational functions

Mathematics Subject Classification 2010: 42C05, 65D32

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Vector-valued inequalities for fractional integrals associated to Jacobi and Laguerre polynomials

Luz Roncal[1]

We study fractional integral operators associated with Jacobi and Laguerre polynomials. Both frameworks can be described in a unified way as follows. The systems of polynomials considered are orthogonal in the corresponding $L^2(X, d\mu)$ spaces, where $X \subset \mathbb{R}$ and μ are suitable measures. Given $\sigma > 0$, the fractional integral of a function $f \in L^2(X, d\mu)$ can be written as an integral operator, which we denote by I_σ, as

$$I_\sigma f(x) = \int_X K_\sigma(x, y) f(y)\, d\mu(y),$$

where the kernel $K_\sigma(x, y)$ is the *potential kernel*. This potential kernel can be expressed in terms of the Poisson kernel or heat kernel defined in the corresponding Jacobi or Laguerre setting, respectively.

We obtain bounds for the potential kernels which are **explicit** in the type parameters of Jacobi or Laguerre polynomials. This fact allows us to get vector-valued extensions for the fractional integrals in both settings. We apply our result in the Jacobi case to analyze fractional integrals on certain compact Riemannian manifolds.

Joint with Ó. Ciaurri and P. R. Stinga.

Keywords: Fractional integral, Jacobi expansions, Laguerre expansions, vector-valued inequalities, analysis on compact Riemannian symmetric spaces of rank one

Mathematics Subject Classification 2010: 42C10, 58J05

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A Riemann-Hilbert problem for sequences of orthogonal Laurent polynomials

Ruymán Cruz-Barroso[1]

In this talk, some important algebraic aspects in the theory of orthogonal Laurent polynomials, such as the three-term recurrence relation, the Christoffel-Darboux or the Liouville-Ostrogradski formulae, are revisited from the Riemann-Hilbert window. These topics are considered for general ordered Laurent polynomial sequences, and not only for the usual "balanced" cases. In addition, the connection with Szegő polynomials (orthogonal polynomials in the unit circle) is explored.

The content is a part of a joint work with Ramón Orive Ángel and Carlos Díaz Mendoza.

Keywords: Riemann-Hilbert problem, orthogonal Laurent polynomials, three-term recurrence relation, Szegő polynomials.

Mathematics Subject Classification 2010: 42C05, 35Q15

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