# Workshop I-Math: Orthogonal Polynomials and Image Processing

**OP&IP-2007: Carmona (Sevilla) October 15-17, 2007**

**Sponsored by**

- Group on Orthogonal Polynomials and Approximation Theory (Univ. de Sevilla)
- Universidad de Sevilla
- Junta de Andalucía (Consejería de Innovación, Ciencia y Empresa)
- Ingenio Mathematica (Programa Consolider-Ingenio 2010, MEC Spain)

The Organizing Committee was:

- Renato Alvarez-Nodarse (Univ. de Sevilla, Spain) (secretary)
- Antonio J. Durán (Univ. de Sevilla, Spain) (chair)

The Scientific Committee was:

- Antonio J. Durán (Univ. de Sevilla, Spain)
- F. Alberto Grünbaum (University of California at Berkeley, USA)

# Program

The schedule of the Workshop was the following:

**Monday 15:**

9:30–10:00: Venue
10:00–11:30: J. Geronimo
11:30–12:00: Coffee break
12:00–13:30: J. Moura
13:30–16:30: Lunch
16:30–17:30: F. Grunbaum
17:30–18:00: Posters/discussion
19:00–20:30: Excursion to Carmona
21:00: Dinner

**Tuesday 16:**

9:30–11:00: J. Moura
11:00–11:30: Coffee break
11:30–13:00: J. Geronimo
13:00–13:30: Posters/discussion
13:30–16:30: Lunch
16:30–17:30: J. Tirao
17:30–18:00: Posters/discussion
21:00: Dinner

**Wednesday 17**:
9:30–11:00: J. Tirao
11:00–11:30: Coffee break
11:30–13:00: F. Grunbaum
13:00–13:30: Closing
13:30–16:30: Lunch

# Abstracts

**Jeff Geronimo** (Dept. of Mathematics, Georgia Institute of Technology, U.S.A.)
**Title:** Orthogonal Polynomials in one and several variables with application to Image processing
**Abstract:** The study of orthogonal polynomials both on the unit circle and the real line is rich in both theory and applications. For recent applications there is the application of the theory of orthogonal polynomials on the real line to the study of the universality properties associated with the distribution of eigenvalues of random matrices, and the use of orthogonal polynomials in the construction of piecewise polynomial, smooth, compactly supported wavelets. As for orthogonal polynomials on the unit circle their use in image processing is well known. These lectures will concentrate on the latter problem. In particular we will consider the autoregressive filter problem and its connection with the orthogonal polynomials. The link with prediction theory will be exposed as well as other classical problems that arise in the theory of functions on the unit disk. We will then consider the autoregressive filter problem in two variables as well as its connections to new results in the theory of functions on the bi-disk.

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**F. Alberto Grünbaum** (Math Dept, University of California, Berkeley, U.S.A.)
**Title:** The reconstruction of images limited in physical and frequency space.
**Abstract:** Certain problems formulated by Claude Shannon in Communication Theory around 1950, and further explored mathematically by D.Slepian, H. Landau and H. Pollack at Bell Labs in the 60's for the case of functions *f(t)* of the time variable *t* play an importante role in areas far removed from this one dimensional setup.

This includes the reconstruction of images in X-ray tomography in the case of "limited angle of views" and the reconstruction of functions defined on part of the surface of the sphere based on the knowledge of some of its coefficients in an expansion in term of spherical harmonics. Both of these problems deal with functions of two variables.

In all these cases ones needs to consider the "singular functions" and the "singular values" of the reconstruction problem. These are given as the eigenfunctions and eigenvalues of a certain integral operator of "time-and-band limiting" built from the characteristics of the specific reconstruction problem. The computation of these quantities is greatly facilitated by the possible existence of a differential operator with simple spectrum that commutes with the integral operator in question.

The existence of such a commuting differential operator hinges on mathematical phenomena that are related to the search of families of orthogonal polynomials that would be joint eigenfunctions of a certain differential operator, or more generally to the search for so called bispectral situations. All of these considerations make sense both in the scalar valued as well as in the matrix valued cases. They also make sense in the case of one or more physical or frequency variables.

I will consider this general problem starting with the simplest and more familiar case of the Fourier transform for scalar functions defined on an interval of the real line (the case of C. Shannon) and revisit the entire problem in a few cases of increasing complexity. This will involve scalar and matrix valued orthogonal polynomials as well as orthogonal polynomials in two variables.

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**José M. F. Moura** (Department of Electrical and Computer Engineering, Carnegie Mellon University, U.S.A.)
**Title:** Algebra, Polynomials, GMRfs: Application to Image Restoration
**Abstract:** Images are examples of random fields - random processes where the indexing parameter is a space variable rather than time. A common application in image processing is image restoration, i.e., filtering out the noise that corrupts the image. With time dependent signals, and under appropriate Gauss-Markov assumptions, the Kalman filter provides a recursive solution to filtering and signal restoration. Unfortunately, Kalman filters cannot be applied to images, or random fields, due to the lack of recursive representations for the fields. In our previous work with Gauss-Markov random fields (GMrf) defined on discrete spaces (lattice), we have been able to derive for the GMrf two equivalent diffusion representations to which we can apply Kalman filtering. In this work, we will consider the extension of these results to GMrfs defined for continuous space. We will show that for one-dimensional GMrfs, from results that describe the covariance of the field as the Green's function of an elliptic operator, we can derive two equivalent recursive representations for the field - a 'forward' and a 'backward' representations - to which we can then apply Kalman filtering. We generalize this result to higher dimensional GMrfs by making use of matrix valued polynomials. We will explain how to derive these results from algebraic reinterpretations of traditional concepts in signal and image processing and from the theory of partial differential equations and matrix valued orthogonal polynomials.

**Juan A. Tirao** (Dpt. Matemáticas, Universidad Nacional de Córdoba, Argentina)
**Title:** Harmonic analysis of matrix valued functions: spherical functions and orthogonal polynomials.
**Abstract:** The case of spherical harmonics plays a role in the imaging of large pieces of the surface of the earth. The examples discussed in the second part of this short course could be of use in other imaging problems.
We will start by recalling the basic facts about harmonic analysis on the sphere $S^2 = SO(3)/SO(2)$ and the role played by the spherical harmonics. We will then restrict our attention to the space of functions that are left invariant under the action of the group $SO(2)$ of rotations of $S^2$ that fix the north pole. This will require the use of zonal spherical functions and Legendre polynomials.
We will then carry out a similar analysis for matrix valued functions defined on the sphere $S^3$, on the complex projective plane $P_2(C) = SU(3)/U(2)$ and finally on the Grassmanian $Gr(4,2;C) = U(4)/U(2) \quad U(2)$ of all complex planes in $C^4$ that contain the origin. We will also discuss the role that matrix valued orthogonal polynomials in one and several variables play in the analysis of each one of the situations discussed above.

# List of Participants

1. Jesús Abderramán, Universidad Politécnica de Madrid

2. Renato Alvarez-Nodarse, Universidad de Sevilla

3. Mirta María Castro Smirnova, Universidad de

4. Óscar Ciaurri, University of La Rioja

5. Santiago Díaz Madrigal, Universidad de Sevilla

6. 5. Manuel Dominguez de la Iglesia, Universidad de Sevilla

7. Antonio Durán Guardeño, Universidad de Sevilla

8. Carmen Escribano, Universidad Politécnica de Madrid

9. Lidia Fernández, University of Granada

10. Jeff Geronimo, Georgia Institute of Technology, USA

11. F. Alberto Grünbaum, University of California at Berkeley, USA

12. Pedro López Rodríguez, Universidad de Sevilla

13. Andrei Martínez-Finkelshtein, Universidad de Almería

14. José M.F. Moura, Carnegie Mellon University, USA

15. Juan Carlos Medem, Universidad de Sevilla

16. Mario Pérez Riera, Universidad de Zaragoza

17. Teresa E. Pérez Fernández, Universidad de Granada

18. José Carlos Petronilho, Universidad de Coimbra

19. Francisco J. Ruiz Blasco, Universidad de Zaragoza

20. M. Asunción Sastre, Universidad Politécnica de Madrid

21. Juan A. Tirao, Universidad Nacional de Córdoba, Argentina

22. Juan Luis Varona, Universidad de La Rioja

23. Alejandro Zarzo, Universidad Politécnica de Madrid