VECTOR MEASURES, INTEGRATION AND APPLICATIONS

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We will deal exclusively with the integration of scalar (i.e. $\mathbb{R}$ or $\mathbb{C}$)-valued functions with respect to *vector measures*. The general theory can be found in [32; 36; 37],[44, Ch.III] and [67; 124], for example. For applications beyond these texts we refer to [38; 66; 80; 102; 117] and the references therein, and the survey articles [33; 68]. Each of these references emphasizes its own preferences, as will be the case with this article. Our aim is to present some theoretical developments over the past 10 years or so (see §1) and to highlight some recent applications. Due to space limitation we restrict the applications to two topics. Namely, the extension of certain operators to their *optimal domain* (see §2) and aspects of *spectral integration* (see §3). The interaction between order and positivity with properties of the integration map of a vector measure (which is defined on a *function space*) will become apparent and plays a central role.

Let Σ be a σ-algebra on a set $\Omega \neq \emptyset$ and E be a locally convex Hausdorff space (briefly, lcHs), over $\mathbb{R}$ or $\mathbb{C}$, with continuous dual space E^*. A σ-additive set function $\nu : \Sigma \to E$ is called a *vector measure*. By the Orlicz-Pettis theorem this is equivalent to the scalar-valued function $x^*\nu : A \mapsto \langle \nu(A), x^* \rangle$ being σ-additive on Σ for each $x^* \in E^*$; its variation measure is denoted by $|x^*\nu|$. A set $A \in \Sigma$ is called *ν-null* if $\nu(B) = 0$ for all $B \in \Sigma$ with $B \subseteq A$. A scalar-valued, Σ-measurable function f on Ω is called *ν-integrable* if

$$f \in L^1(x^*\nu), \qquad x^* \in E^*, \tag{0.1}$$

and, for each $A \in \Sigma$, there exists $x_A \in E$ such that

$$\langle x_A, x^* \rangle = \int_A f \, dx^*\nu, \qquad x^* \in E^*. \tag{0.2}$$

We denote x_A by $\int_A f \, d\nu$. Two ν-integrable functions are identified if they differ on a ν-null set. Then $L^1(\nu)$ denotes the linear space of all (equivalence classes of) ν-integrable functions (modulo ν-a.e.).

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Let Q denote a family of continuous seminorms determining the topology of E. Each $q \in Q$ induces a seminorm in $L^1(\nu)$ via

$$\|f\|_q := \sup_{x^* \in U_q^\circ} \int_\Omega |f| \, d|x^*\nu|, \qquad f \in L^1(\nu), \tag{0.3}$$

where $U_q^\circ \subseteq E^*$ is the polar of $q^{-1}([0,1])$. The Σ-simple functions are always dense in the lcHs $L^1(\nu)$. For E a Banach space, we have the single norm

$$\|f\|_{L^1(\nu)} = \sup_{\|x^*\| \le 1} \int_\Omega |f| \, d|x^*\nu|, \qquad f \in L^1(\nu). \tag{0.4}$$

Whenever E is a Fréchet space, $L^1(\nu)$ is metrizable and complete. We point out that (0.3) is a *lattice seminorm*, i.e. $\|f\|_q \le \|g\|_q$ whenever $f, g \in L^1(\nu)$ satisfy $|f| \le |g|$. Actually, $L^1(\nu)$ is also *solid*, i.e. $f \in L^1(\nu)$ whenever $|f| \le |g|$ with f measurable and $g \in L^1(\nu)$. If we wish to stress that we are working over $\mathbb{R}$ or $\mathbb{C}$, then we write $L^1_{\mathbb{R}}(\nu)$ or $L^1_{\mathbb{C}}(\nu)$, resp. Of course, $L_{\mathbb{C}}(\nu) = L_{\mathbb{R}}(\nu) + iL_{\mathbb{R}}(\nu)$ is the complexification of $L^1_{\mathbb{R}}(\nu)$, with the order in the positive cone $L^1(\nu)^+$ of $L^1(\nu)$ being that defined pointwise ν-a.e. on Ω. The dominated convergence theorem ensures that the topology in the (lc-lattice $=$) lc-Riesz space $L^1(\nu)$ has the *σ-Lebesgue property* (also called *σ-order continuity* of the norm if E is Banach), i.e. if $\{f_n\} \subseteq L^1(\nu)$ satisfies $f_n \downarrow 0$ with respect to the order, then $\lim_{n\to\infty} f_n = 0$ in the topology of $L^1(\nu)$. Moreover, χ_Ω is a *weak order unit* in $L^1(\nu)$; see [3] for the definition.

If a Σ-measurable function f satisfies only (0.1), then (0.3) is still finite for each $q \in Q$, [128]; we then say that f is *weakly ν-integrable* and denote the space of all (classes of) such functions by $L^1_w(\nu)$. Equipped with the seminorms $\{\|\cdot\|_q : q \in Q\}$, this is a *lc-lattice* (Fréchet whenever E is Fréchet) containing $L^1(\nu)$ as a closed subspace. If E does not contain an isomorphic copy of c_0, then necessarily $L^1_w(\nu) = L^1(\nu)$; see [67, II 5 Theorem 1] and [73, Theorem 5.1]. For all of the above basic facts (and others) concerning $L^1_{\mathbb{R}}(\nu)$ we refer to [67] and for $L^1_{\mathbb{C}}(\nu)$ to [50], for example. If E is not complete, then various subtleties may arise in passing from the case of $\mathbb{R}$ to $\mathbb{C}$, [59; 91; 119].

Given $1 \le p < \infty$, a Σ-measurable function f belongs to $L^p(\nu)$ if $|f|^p \in L^1(\nu)$. When E is a Banach space, then $L^p(\nu)$ is a *p-convex Banach lattice* relative to the norm

$$\|f\|_{L^p(\nu)} := \sup_{\|x^*\| \le 1} \left(\int_\Omega |f|^p \, d|x^*\nu| \right)^{1/p}, \qquad f \in L^p(\nu), \tag{0.5}$$

and satisfies $L^p(\nu) \subseteq L^1(\nu)$ continuously. Such spaces and operators defined in them have recently been studied in some detail; see [55; 56;

122], for example, and the references therein. Of course, the spaces $L^p_w(\nu)$ can also be defined in the obvious way (i.e. by requiring $|f|^p \in L^1_w(\nu)$), [56]. More generally, a study of Orlicz spaces with respect to a vector measure, which stems from a detailed study of the Banach function subspaces of $L^1(\nu)$, has been made in [27].

Of central importance to this article will be the *integration operator* $I_\nu : L^1(\nu) \to E$ defined by $f \mapsto \int_\Omega f\, d\nu$. For brevity, we also write $\int f\, d\nu$ for $\int_\Omega f\, d\nu$. According to (0.2) and (0.3) we have

$$q\left(\int_\Omega f\, d\nu\right) = \sup_{x^* \in U_q^\circ} |\langle \int_\Omega f\, d\nu, x^*\rangle| \le \|f\|_q, \qquad f \in L^1(\nu),$$

for each $q \in Q$. This shows that I_ν is continuous, with $\|I_\nu\| \le 1$ if E is a Banach space. Given a lc-lattice E, a vector measure $\nu : \Sigma \to E$ is called *positive* if it takes its values in the positive cone E^+ of E. In this case, it is an easy approximation argument using the denseness of the Σ-simple functions in $L^1(\nu)$ to see that I_ν is a *positive operator*, i.e. $I_\nu(L^1(\nu)^+) \subseteq E^+$. For recent aspects of the theory of vector measures and integration in a Banach space E see [16; 17; 18; 19] and for E a Fréchet space or lattice, we refer to [8; 31; 48; 49; 51; 52; 53; 54; 125; 126; 127], and the references therein.

1. Representation Theorems

When E is a Banach space and ν is an E-valued measure, the correct framework for interpreting both $L^1(\nu)$ and $L^1_w(\nu)$ is that of *Banach function spaces* (briefly, B.f.s.). Let $(\Omega, \Sigma, \lambda)$ be a σ-finite measure space, $\mathcal{M}$ be the space of all Σ-measurable functions on Ω (functions equal λ-a.e. are identified), and $\mathcal{M}^+$ be the cone of those elements of $\mathcal{M}$ which are non-negative λ-a.e. A *function norm* is a map $\rho\colon \mathcal{M}^+ \to [0,\infty]$ satisfying

(a) $\rho(f) = 0$ iff $f = 0$ λ-a.e.,
$\rho(af) = a\rho(f)$ for every $a \ge 0$,
$\rho(f+g) \le \rho(f) + \rho(g)$ for all $f, g \in \mathcal{M}^+$,

(b) If $f, g \in \mathcal{M}^+$ and $f \le g$ λ-a.e., then $\rho(f) \le \rho(g)$.

The function space L_ρ is defined as the set of all $f \in \mathcal{M}$ satisfying $\rho(|f|) < \infty$; it is a linear space and ρ is a norm. Moreover, whenever ρ has the Riesz-Rischer property (c.f. [136, Ch.15]) the space L_ρ is a Banach lattice for the λ-a.e. order and it is always an an ideal of measurable functions, that is, if $f \in L_\rho$ and $g \in \mathcal{M}$ satisfies $|g| \le |f|$ λ-a.e., then $g \in L_\rho$. The *associate space* $L_{\rho'}$ of L_ρ is generated by the function norm $\rho'(g) := \sup\{\int |fg|\, d\lambda : \rho(f) \le 1, f \in \mathcal{M}^+\}$. We also denote $L_{\rho'}$ by L'_ρ. If $g \in L'_\rho$ and $G(f) := \int fg\, d\lambda$ for every $f \in L_\rho$, then

$G \in L_\rho^*$ and $\|G\| = \rho'(g)$. In this sense, L'_ρ is identified with a closed subspace of L_ρ^*. Applying the same procedure to L'_ρ, we obtain the second associate space L''_ρ. The B.f.s. L_ρ satisfies the *Fatou property* if $0 \le f_n \uparrow f$ in $\mathcal{M}^+$ implies that $\rho(f_n) \uparrow \rho(f)$. The *Lorentz function norm* ρ_L associated to any given function norm ρ is defined by

$$\rho_L(f) := \inf\{\lim \rho(f_n) : 0 \le f_n \uparrow f \text{ with } f_n \in \mathcal{M}^+\}.$$

Then ρ_L is the largest norm majorized by ρ and having the Fatou property, [136, Ch.15, §66]. It follows that L_{ρ_L} is the *minimal* B.f.s. (over λ) with the Fatou property and continuously containing (with norm ≤ 1) L_ρ. Since $\rho_L = \rho''$ [136, Ch.15, §71, Theorem 2], we see that L''_ρ is the minimal B.f.s. (over λ) with the Fatou property and continuously containing (with norm ≤ 1) L_ρ. The *order continuous part* $(L_\rho)_a$ of a B.f.s. L_ρ consists of all $f \in L_\rho$ such that $\rho(f_n) \downarrow 0$ whenever $\{f_n\} \subseteq L_\rho^+$ satisfies $|f| \ge f_n \downarrow 0$ (equivalently, increasing sequences order bounded by $|f|$ are norm convergent, [136, Ch.15, §72, Theorem 2]). B.f.s.' were studied by Luxemburg and Zaanen; see [75; 76; 77; 78; 79] and [136, Ch.15]. Caution should be taken since some authors consider different definitions of B.f.s.' which are more restrictive; see [6, Definition I.1.1] and [74, Definition 1.b.17].

It was observed in [17, Theorem 1] that $L^1(\nu)$ is a σ-order continuous (briefly, σ-o.c.) B.f.s. with weak order unit with respect to the measure space $(\Omega, \Sigma, \lambda)$, where λ is a Rybakov control measure for the vector measure $\nu : \Sigma \to E$, that is, a finite measure of the form $\lambda = |x_0^*\nu|$ for some suitable $x_0^* \in B_{E^*}$ (the closed unit ball of E^*) such that λ and ν have the same null sets; see [32, Ch.IX, §2, Theorem 2]. The crucial property is σ-order continuity, which leads to the converse result (c.f. Theorem 1.1 below); this is the main representation theorem for the class of spaces $L^1(\nu)$, [17, Theorem 8]. Note that in Banach lattices, σ-order continuity and σ-Dedekind completness (which is satisfied by $L^1(\nu)$) is equivalent to order continuity (i.e. $\|x_\tau\| \downarrow 0$ whenever $\{x_\tau\}$ decreases to zero in order).

Theorem 1.1. *Let E be any Banach lattice with o.c. norm and possessing a weak order unit. Then there exists a (E^+-valued) vector measure ν such that E is order and isometrically isomorphic to $L^1(\nu)$.*

This theorem characterizes the spaces $L^1(\nu)$, for Banach space-valued measures, and explains the diversity of spaces arising as $L^1(\nu)$. For example, for a finite measure space $(\Omega, \Sigma, \lambda)$, $1 \le p < \infty$, and the vector measure $A \mapsto \nu_p(A) := \chi_A \in L^p(\lambda)$ on Σ, we obtain $L^1(\nu_p) = L^p(\lambda)$.

The theory of integrating scalar-valued functions with respect to a vector measure defined on a σ-algebra can be extended to vector measures defined on δ-rings; [73; 81; 82]. A study of the corresponding space of integrable functions has been undertaken in [28]. In this context, Theorem 1.1 can be generalized as follows, [16, pp.22-23].

Theorem 1.2. *Let E be any Banach lattice with o.c. norm. Then there exists an E-valued vector measure ν defined on a δ-ring such that E is order and isometrically isomorphic to $L^1(\nu)$.*

A version of these representation theorems in a more general setting is also known, [41, Proposition 2.4(vi)]. For the definition of a spectral measure, see Section 3.

Theorem 1.3. *Let E be a Dedekind complete, complex Riesz space with locally solid, Lebesgue topology. Assume E is quasicomplete and has a weak order unit $e \geq 0$ and that the space of continuous linear operators in E is sequentially complete for the strong operator topology. Then there exists a closed, equicontinuous $\mathcal{L}(E)$-valued spectral measure for which e is a cyclic vector and such that the integration map Pe is a topological and Riesz homomorphism of $L^1(Pe)$ onto E.*

What is the connection between the spaces $L^1(\nu)$ and $L^1_w(\nu)$? For the function norm (relative to $\lambda = |x_0^*\nu|$ as above) given by

$$\rho_w(f) := \sup_{x^* \in B_{E^*}} \int_\Omega |f| \, d|x^*\nu|, \qquad f \in \mathcal{M},$$

we have $L_{\rho_w} = L^1_w(\nu)$. So: what is the connection between the B.f.s.' $L^1(\nu)$ and $L^1_w(\nu)$? In this regard the role of the Fatou property is relevant, [23, Propositions 2.1, 2.3 and 2.4].

Theorem 1.4. *Let ν be any vector measure.*

(a) *The B.f.s. $L^1_w(\nu)$ has the Fatou property.*
(b) *$L^1(\nu)'' = L^1_w(\nu)$.*
(c) *$L^1(\nu)$ has the Fatou property iff $L^1_w(\nu)$ has o.c.-norm.*

The answer to the question above can now be given. Since $L^1(\nu)$ has o.c.-norm and the Σ-simple functions are dense, it can be verified that $(L^1_w(\nu))_a = L^1(\nu)$. So, $L^1(\nu)$ is the *maximal* B.f.s. inside $L^1_w(\nu)$ (with the same norm) which has o.c.-norm. Since the Lorentz function norm ρ_L is the largest norm majorized by ρ and having the Fatou property, [136, Ch.15, §66], it follows that $L^1_w(\nu)$ is the *minimal* B.f.s. (over λ) with the Fatou property and continuously containing (with norm ≤ 1) $L^1(\nu)$. Accordingly, $L^1_w(\nu)$ can be interpreted as the "Fatou completion" of $L^1(\nu)$. Statement (c) in Theorem 1.4 now follows: if

$L^1(\nu)$ has the Fatou property, then the minimal property of $L^1_w(\nu)$ forces $L^1(\nu) = L^1_w(\nu)$; on the other hand, if $L^1_w(\nu)$ has o.c.-norm, then the maximal property of $L^1(\nu)$ forces $L^1_w(\nu) = L^1(\nu)$.

Theorem 1.5 below characterizes all Banach lattices which arise as $L^1_w(\nu)$ for some vector measure ν, [23, Theorem 2.5]. Note what we have called the Fatou property, technically speaking, should be called the σ-Fatou property but, since B.f.s.' are super Dedekind complete and $\mathcal{M}$ is order separable, there is no distinction between using increasing sequences or increasing nets, [137, Theorem 112.3].

Theorem 1.5. *Let E be any Banach lattice with the σ-Fatou property and possessing a weak order unit which belongs to E_a. Then there exists a (E^+_a-valued) vector measure ν such that E is order and isometrically isomorphic to $L^1_w(\nu)$.*

For E a B.f.s., the previous result gives the following

Theorem 1.6. *Let L_ρ be a B.f.s. over a finite measure space $(\Omega, \Sigma, \lambda)$ such that L_ρ has the Fatou property and $\chi_\Omega \in (L_\rho)_a$. Then L_ρ is order and isometrically isomorphic to $L^1_w(\nu)$ for some ν.*

For p-convex Banach lattices with $1 \le p < \infty$ (see [74, Definition 1.d.3]) the following extension of Theorem 1.1 holds, [56, Theorem 2.4].

Theorem 1.7. *Let E be a p-convex Banach lattice with o.c. norm and a weak order unit. Then there exists a vector measure ν such that E is order isomorphic to $L^1(\nu)$.*

For $1 \le p < \infty$, the space $L^p_w(\nu)$ is generated by the function norm

$$\rho^p_w(f) := \sup_{x^* \in B_{E^*}} \left(\int_\Omega |f|^p \, d|x^*\nu| \right)^{1/p}, \qquad f \in \mathcal{M}.$$

The result corresponding to Theorem 1.4, now relating the spaces $L^p(\nu)$ and $L^p_w(\nu)$ is also known, [26, Propositions 1, 2 and 4]. For $E = L^p_w(\nu)$, we know that E is p-convex, has the σ-Fatou property, $E_a = L^p(\nu)$ and χ_Ω is a weak order unit for E which belongs to E_a. These properties of $L^p_w(\nu)$ characterize a large class of abstract Banach lattices, [26, Theorem 4].

Theorem 1.8. *Let $1 \le p < \infty$ and E be any p-convex Banach lattice with the σ-Fatou property and possessing a weak order unit which belongs to E_a. Then there exists an E_a-valued vector measure ν such that E is Banach lattice isomorphic to $L^p_w(\nu)$.*

We conclude with a different kind of representation result, not of a space, but of an operator. The *variation measure* of a Banach space-valued vector measure ν, denoted by $|\nu|$, can be defined via the "partition process" as for scalar measures; see [32, pp.2-3]. It turns out that always $L^1(|\nu|) \subseteq L^1(\nu)$ with a continuous inclusion, [73]. For the notion of *Bochner integrals* we refer to [32, Ch. II].

Theorem 1.9. *Let E be a Banach space and $\nu : \Sigma \to E$ be a vector measure with finite variation (i.e. $|\nu|(\Omega) < \infty$). The integration map $I_\nu : L^1(\nu) \to E$ is compact iff ν possesses an E-valued, Bochner $|\nu|$-integrable Radon-Nikodým derivative $G = d\nu/d|\nu|$ which has $|\nu|$-essentially relatively compact range in E.*

In this case, $L^1(\nu) = L^1(|\nu|)$ and (with Bochner integrals) we have

$$I_\nu f = \int_\Omega fG \, d|\nu|, \qquad f \in L^1(\nu).$$

Remark 1.10. (i) This result occurs in [96, Theorem 1].

(ii) It is also true that if I_ν is compact, then ν has finite variation, [96, Theorem 4]. Examples of vector measures which do not have finite variation arise via a *Pettis integrable* density $G : \Omega \to E$ (see [32, Ch. II, §3] for the definition), i.e. $\nu(A) := \int_A G \, d\lambda$, where $\lambda : \Sigma \to [0, \infty]$ is an infinite measure; see [46, Proposition 5.6(iv)] where it is shown that $|\nu|$ is σ-finite. More precisely, if G is strongly measurable, [32, p.41], and Pettis λ-integrable (but *not* Bochner λ-integrable), then ν has σ-finite but, not finite, variation. For the existence of such functions G on $\Omega = [0, \infty)$ for Lebesgue measure λ, see the proof of [120, Theorem 3.3]. A characterization of vector measures with σ-finite variation occurs in [121, Theorem 2.4]. In every infinite dimensional space E there also exist vector measures with infinite but, not σ-finite, variation; they can even be chosen to have relatively compact range, [132, p.90]. In particular, relative compactness of the range of ν does not suffice for I_ν to be compact. For concrete examples of ν (arising in classical analysis) which fail to have σ-finite variation we refer to [87, Lemma 2.1] and [99, Proposition 4.1], for example.

(iii) Let $\dim(E) = \infty$. Then there exists an E-valued measure ν with $|\nu|(\Omega) < \infty$, the range of ν is not contained in any finite dimensional subspace of E, and I_ν is compact, [96, Theorem 2]. There also exists an E-valued measure μ with $|\mu|(\Omega) < \infty$ satisfying $L^1(|\mu|) = L^1(\mu)$ and having an E-valued Bochner $|\mu|$-integrable Radon-Nikodým derivative $d\mu/d|\mu|$ such that I_μ is not compact [96, Theorem 3].

(iv) For $1 < p < \infty$, the compactness properties of I_ν, restricted to $L^p_{\mathbb{R}}(\nu) \subseteq L^1_{\mathbb{R}}(\nu)$ and $L^p_w(\nu)_{\mathbb{R}} \subseteq L^1_{\mathbb{R}}(\nu)$, are studied in [56; 122].

(v) An extension of Theorem 1.9 to Fréchet spaces E occurs in [97]. Given extra properties of E, more can be said. For instance, if E is Fréchet-Montel, then $I_\nu : L^1(\nu) \to E$ is compact iff the Fréchet lattice $L^1(\nu)$ is order and topologically isomorphic to a Banach AL-lattice, [98, Theorem 2]. Nuclearity of E also has some consequences, [98, Theorem 1].

2. Optimal domains

Let X be a B.f.s. over a finite measure space $(\Omega, \Sigma, \lambda)$, E be a Banach space and $T\colon X \to E$ be a linear operator. There arise situations when T has a natural extension (still with values in E) to a larger space Y into which X is continuously embedded. This is the case for the Riesz representation theorem: a positive linear operator $\Lambda\colon C(K) \to \mathbb{C}$ can be extended to the space $L^1(\lambda)$, where λ is a scalar measure associated to Λ. Under certain conditions, we associate to the operator T a vector measure ν_T with values in E. We will say that the operator T is λ-*determined* if the additive set function (with values in E)

$$\nu_T : A \mapsto T(\chi_A), \qquad A \in \Sigma$$

has the same null sets as λ. For the next result see [20, Theorem 3.1].

Theorem 2.1. *Let X be a B.f.s. over a finite measure space $(\Omega, \Sigma, \lambda)$, E be a Banach space and $T\colon X \to E$ be a λ-determined linear operator with the property that $Tf_n \to Tf$ weakly in E whenever $\{f_n\} \subset X$ is a positive sequence increasing λ-a.e to $f \in X$. Then the measure ν_T is countably additive, X is continuously embedded in $L^1(\nu_T)$ and the integration operator from $L^1(\nu_T)$ into E extends T.*

Remark 2.2. (i) The assumptions of Theorem 2.1 hold if X has order continuous norm and T is continuous and linear. They also hold for $X = L^\infty(\lambda)$ and T weak*-to-weak continuous. Continuity of T alone does not suffice in general (e.g. the identity operator on $L^\infty([0,1])$). The result can be extended to σ-finite measure spaces, or even general measure spaces provided E contains a weak order unit, that is, a function $\varphi > 0$, λ-a.e. In this case the measure ν is defined by $\nu_T(A) := T(\varphi\chi_A)$, $A \in \Sigma$, and the embedding from E into $L^1(\nu_T)$ is $f \mapsto f/\varphi$.

(ii) Theorem 2.1 provides an integral representation for certain operators via integration with respect to a vector measure, even in cases where the Bochner or Pettis integrals do not exist. For instance, the fractional integral of order α, $0 < \alpha < 1$, of a function f at a point

$x \in [0,1]$ is given by

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{f(t)}{|x-t|^{1-\alpha}}\, dt$$

whenever it is defined. We can consider I_α as an operator $I_\alpha \colon L^\infty([0,1]) \to L^p([0,1])$. By Theorem 2.1, $I_\alpha(f) = \int f\, d\nu_p$, where ν_p denotes the measure given by $\nu_p(A)(x) = \int_A |x-y|^{\alpha-1}\, dy \in L^p([0,1])$. This can be done for any $1 \le p \le \infty$. However, unless $(1-\alpha)p < 1$, there is no Bochner or Pettis integrable function $G\colon [0,1] \to L^p([0,1])$ such that $I_\alpha(f) = \int_{[0,1]} f(t)G(t)\, dt$, [20, Remark 3.5].

A basic problem is to identify the optimal space to which the operator T can be extended, within a particular class of spaces, but keeping the codomain space of T fixed. This is sometimes considered within the theory of integral operators; see [4; 71; 88; 129; 130], for example, and the references therein. We will denote by $[T,E]$ the maximal B.f.s. (containing X) to which T can be extended as a continuous linear operator, still with values in E. This maximality is to be understood in the following sense. There is a continuous linear extension of T (which we still denote by T) $T\colon [T,E] \to E$, and if T has a continuous, linear extension $\tilde{T}\colon Y \to E$, where Y is a B.f.s. containing X, then Y is continuously embedded in $[T,E]$, and the extended operator T coincides with $\tilde{T}$ on Y. The space $[T,E]$ is then the *optimal lattice domain* for T. If we consider the class of B.f.s.' with order continuous norm (briefly, o.c.), then we have the space $[T,E]_o$, which is the *o.c. optimal lattice domain* for T. Theorem 2.1 shows that the space $L^1(\nu_T)$ is the o.c. optimal lattice domain for T.

We identify situations in which $[T,X] = L^1(\nu_T)$; this has the advantage that the properties of ν_T and E, needed in determining the space $L^1(\nu_T)$ hence, also $[T,E]$, are well understood. As we will see, this procedure for identifying optimal domains is extremely fruitful.

Let $K\colon [0,1]\times[0,1] \to [0,\infty)$ be a measurable function. We associate to K an operator T via the formula

$$Tf(x) := \int_0^1 f(y)K(x,y)\, dy, \qquad x \in [0,1], \tag{2.1}$$

for any function f for which it is meaningful to do so for m-a.e. $x \in [0,1]$, where m is Lebesgue measure in [0,1]. We will say that the kernel K is *admissible* if it satisfies the following three conditions: (i) for every $x \in [0,1]$, the function $K_x\colon y \mapsto K(x,y), y \in [0,1]$, is Lebesgue integrable in [0,1]; (ii) $\int_0^1 K_y(x)\, dx > 0$ for m-a.e. $y \in [0,1]$ where, for every $y \in [0,1]$, the function K_y is defined by $x \mapsto K(x,y)$, for

$x \in [0,1]$; and (iii) $K_{x_n} \to K_{x_0}$ weakly in $L^1([0,1])$ whenever $x_0 \in [0,1]$ and $x_n \to x_0$. These conditions guarantee that

$$\nu(A)(\cdot) := \int_A K(\cdot\,, y)\,dy, \qquad A \in \mathcal{B},$$

($\mathcal{B}$ is the Borel σ-algebra of [0,1]) is a $C([0,1])$-valued, σ-additive measure, which is m-determined, [20, Proposition 4.1].

Let E be a B.f.s. over $([0,1],\mathcal{B},m)$ for which $L^\infty([0,1]) \subseteq E \subseteq L^1([0,1])$. Under the above conditions on K, we have $T\colon L^\infty([0,1]) \to E$ continuously. It turns out that $[T,E] = \{f : T|f| \in E\}$, [20, Proposition 5.2], and $\|f\|_{[T,E]} := \|\,T|f|\,\|_E$ is a complete function norm in $[T,E]$. Since $C([0,1])$ is continuously embedded in E, the measure ν is also E-valued and σ-additive with $\nu(A) = T(\chi_A)$ for $A \in \mathcal{B}$. We denote it by ν_E in this case. Moreover, because $K \geq 0$, it is clear that $T\colon E \to E$ is a *positive operator* and that ν_E takes its values in E^+.

The relationships between the three B.f.s.' associated to K and E, namely, $L^1(\nu_E)$, $L^1_w(\nu_E)$ and $[T,E]$, are precise, [23, p.199].

Theorem 2.3. *Let K be a non-negative admissible kernel and E be a B.f.s. satisfying the above conditions. The following inclusions hold:*

$$L^1(\nu_E) \subseteq [T,E] \subseteq [T,E]'' = L^1_w(\nu_E) \subseteq [T,E'']. \tag{2.2}$$

The first inclusion is an isometric imbedding, and the norms of the spaces $[T,E]''$ and $L^1_w(\nu_E)$ coincide.

Remark 2.4. (i) The inclusions in (2.2) can be strict; see [20, Remark 5.3] and [23, Example 3.4].

(ii) The optimal domain $[T,E]$ has the Fatou property iff $[T,E] = L^1_w(\nu_E)$, [23, p.199].

(iii) If E' is a norming subspace of E^*, then all inclusions in (2.2) are isometric imbeddings and $L^1_w(\nu_E) = [T,E'']$; see Corollary 3.5 and Theorem 3.6 of [23].

(iv) If E has o.c. norm, then $L^1(\nu_E) = [T,E]$ and

$$L^1(\nu_E) = [T,E] \subseteq [T,E]'' = L^1_w(\nu_E) = [T,E''],$$

with the imbedding $[T,E] \subseteq L^1_w(\nu_E)$ isometric, [23, Corollary 3.7].

(v) If E has the Fatou property, then $L^1_w(\nu_E) = [T,E]$ and

$$L^1(\nu_E) \subseteq [T,E] = [T,E]'' = L^1_w(\nu_E) = [T,E''];$$

see [23, Corollary 3.7].

(vi) If E is weakly sequentially complete, then all spaces in (2.2) are equal, [23, Corollary 3.7].

An interesting result concerning the relationships between the spaces $L^1(\nu_E)$, $L^1_w(\nu_E)$ and $[T,E]$ is the following one, [23, Proposition 3.12]. Recall, for a B.f.s. E, that E_a (c.f. Section 1) is the *o.c. part* of E and E_b is the closure of the $\mathcal{B}$-simple functions in E.

Theorem 2.5. *Let Y be any one of $L^1(\nu_E)$, $L^1_w(\nu_E)$ or $[T,E]$. Then*

$$Y_b = Y_a = L^1(\nu_E).$$

Additional information on the optimal domain $[T,E]$ is available in the case when the B.f.s. E is rearrangement invariant (briefly, r.i.). Recall that a B.f.s. E is r.i. if it satisfies the Fatou property and $f \in E$ implies that $g \in E$ with $\|g\| = \|f\|$ whenever g and f are equimeasurable, [6, II.4.1]. Every r.i. space E on [0,1] is an interpolation space between the spaces $L^1([0,1])$ and $L^\infty([0,1])$, arising via the the K-functional of Peetre (as $E = (L^1, L^\infty)_\rho$ for a suitable r.i. norm ρ); see [6, §V.1]. It turns out, under certain conditions, that the optimal domain $[T,E]$ is an interpolation space between the optimal domains $[T, L^1([0,1])]$ and $[T, L^\infty([0,1])]$ (both of these spaces being weighted L^1-spaces) in the same way that E is an interpolation space between $L^1([0,1])$ and $L^\infty([0,1])$ (by a technical result of Gagliardo, we can substitute here $L^\infty([0,1])$ with $C([0,1])$). Let us be more precise.

Theorem 2.6. *Let K be a non-negative admissible kernel.*

(a) *Let ν be the associated $C([0,1])$-valued measure. Then we have $[T, C([0,1])] = L^1(\nu)$.*
(b) *If, in addition, K is non-decreasing (i.e. $K_{x_1} \le K_{x_2}$ a.e. on $[0,1]$ whenever $0 \le x_1 \le x_2 \le 1$), then $[T, C([0,1])] = L^1_\xi$ where the weight $\xi(y) := K(1,y)$.*
(c) *Let ν_{L^1} denote the measure ν considered as being $L^1([0,1])$-valued. Then $[T, L^1([0,1])] = L^1(\nu_{L^1}) = L^1_\omega$, where the weight $\omega(y) := \int_0^1 K(x,y)\,dx$.*

This identification (see Propositions 5.1 and 5.4 of [20]) makes it possible, with the aid of interpolations techniques, to obtain the following result, [20, Proposition 5.5 & Theorem 5.11].

Theorem 2.7. *Let K be a non-negative admissible kernel and $E = (L^1, L^\infty)_\rho$ be a r.i. B.f.s. on $[0,1]$.*

(a) *The space $(L^1_\omega, L^1(\nu))_\rho$ is continuously embedded in $[T,X]$.*
(b) *If K is non-decreasing with the property that there exists a constant $\beta > 0$ such that, for every $t > 0$ and every $y \in [0,1]$,*

$$\int_{\max\{0,1-t\}}^1 K(x,y)\,dx \ge \beta \cdot \min\left\{ \int_0^1 K(x,y)dx;\ t \cdot K(1,y) \right\}, \qquad (*)$$

then, with equivalence of norms,

$$[T,E]=(L^1_\omega,L^1_\xi)_\rho.$$

(c) *If, in addition, E has order continuous norm, then* $[T,E]=(L^1_\omega,L^1_\xi)_\rho=L^1(\nu_E)$.

Many interesting kernels satisfying the above properties occur. For example, the classical Volterra operator given by the kernel $K(x,y)=\chi_\Delta(x,y)$, where $\Delta=\{(x,y)\in[0,1]^2\colon 0\le y\le x\}$, is non-decreasing and satisfies condition $(*)$ with $\beta=1$. There is a corresponding result for non-increasing kernels, [20, Theorem 5.12], an example of which is given by the kernel arising from nilpotent left translation semigroups; see [20, Example 4.4 and Remark 5.14].

The extension of the previous results to kernel operators defined for functions on $[0,\infty)$ requires a consideration of optimal domains for operators defined on B.f.s.' over $[0,\infty)$. For this, the required tool is the theory of L^1-spaces for vector measures on δ-rings. Such a study is made in [29]; for applications to the Hardy operator, see [30].

The above results on optimal domains for kernel operators can be applied to the study of refinements of the classical Sobolev inequality. This inequality, valid for differentiable functions f on a bounded domain Ω in $\mathbb{R}^n$ with $n\ge 2$, states that

$$\|f\|_{L^q(\Omega)}\le C\|\,|\nabla f|\,\|_{L^p(\Omega)},\qquad f\in C^1_0(\Omega),\tag{2.3}$$

where $1<p<n$, $q:=np/(n-p)$ and $C>0$ depends only on p, n. Edmunds, Kerman and Pick studied the optimal domain problem, for the inequality (2.3), within the class of r.i. spaces, [47]. They consider r.i. spaces E and F on $[0,1]$ and a generalized Sobolev inequality

$$\|f^*\|_E\le C\||\nabla f|^*\|_F,\qquad f\in C^1_0(\Omega),\tag{2.4}$$

where f^* and $|\nabla f|^*$ are, respectively, the decreasing rearrangements of f and the norm of its gradient $|\nabla f|$. They show (for $|\Omega|=1$) that (2.4) is equivalent to boundedness of the kernel operator T associated with Sobolev's inequality, namely

$$Tf(t)=\int_t^1 f(s)s^{(1/n)-1}\,ds,\qquad t\in[0,1],\tag{2.5}$$

acting between the r.i. spaces E and F, that is, $\|Tf\|_E\le K\|f\|_F$, [47, Theorem 6.1]. Since the kernel in (2.5) satisfies the conditions of Theorem 2.7(b) it follows, for a r.i. space $E=(L^1,L^\infty)_\rho$, that the optimal domain $[T,E]$ can be identified as the interpolation space

$$[T,E]=\big(L^1(s^{1/n}ds),L^1(s^{(1/n)-1}ds)\big)_\varrho\,,$$

since $[T, L^\infty([0,1])] = L^1(s^{(1/n)-1}ds)$ and $[T, L^1([0,1])] = L^1(s^{1/n}ds)$, [21, Proposition 2.1(d) and Corollary 4.3].

For the kernel operator associated to Sobolev's inequality a thorough study of the optimal domains has been made; see the following result, [21, Proposition 3.1(a), Theorem 4.2 and Corollary 4.3].

Theorem 2.8. *The optimal domain* $[T, E]$ *is order isomorphic to an* AL*-space if and only if* E *is a Lorentz* Λ*-space, in which case*

$$[T, E] = L^1(\nu_E) = L^1(|\nu_E|).$$

Moreover, the variation measure $|\nu_E|$ *is given by*

$$|\nu_E|(A) = \int_A s^{(1/n)-1}\varphi_E(s)\, ds, \qquad A \in \mathcal{B},$$

with $\varphi_E(s) := \|\chi_{[0,s]}\|_E$ *the fundamental function of the r.i. space* E*.*

The question of whether or not the spaces $[T, E]$ are r.i. is an important one. The answer is given in [24, Theorem 3.4].

Theorem 2.9. *Let* E *be a r.i. space on* $[0,1]$*. The optimal domain* $[T, E]$ *is itself r.i. if and only if* E *is the Lorentz space* $L^{n',1}([0,1])$*.*

This result focuses the investigation on the *largest* r.i. space continuously contained in the optimal domain $[T, E]$, denoted by $[T, E]^{ri}$. An example illustrates the importance of this issue. For $E = L^p([0,1])$ and $n' < p < \infty$, the optimal r.i. domain $[T, L^p]^{ri}$ is the Lorentz $L^{p,q}$-space $L^{p_0,p}([0,1])$, where $p_0 := np/(n+p)$, [108, Theorem 3.20]. Note that $p = np_0/(n-p_0)$ is precisely the exponent corresponding to p_0 in the classical Sobolev inequality (2.3):

$$\|f\|_p \le C \, \| \, |\nabla f| \, \|_{p_0}.$$

Hence, Sobolev's inequality is actually sharpened, since $[T, L^p]^{ri} = L^{p_0,p}([0,1])$ implies that

$$\|f\|_p \le C \, \| \, |\nabla f| \, \|_{p_0,p},$$

with $\| \, |\nabla f| \, \|_{p_0,p} \le \| \, |\nabla f| \, \|_{p_0,p_0} = \| \, |\nabla f| \, \|_{p_0}$ (as $p > p_0$). Moreover, this sharpening is optimal within the class of r.i. norms.

The following result identifies $[T, E]^{ri}$ for certain classes of r.i. spaces E, [21, Proposition 4.7, Theorem 5.7 and Theorem 5.11].

Theorem 2.10. *Let* E *a r.i. B.f.s. on* $[0,1]$*.*

(a) *Suppose that* φ_E *is* $(1/n')$*-quasiconcave. Then, for* $\Theta(t) := \int_0^t s^{(1/n)-1}\varphi_E(s)\, ds$*, the Lorentz space* Λ_Θ *is the largest r.i. space inside* $L^1(|\nu_E|)$*.*

(b) *Let E be a Marcinkiewicz space M_φ with φ satisfying $(1/n) < \gamma_\varphi \le \delta_\varphi < 1$. Then the largest r.i. space inside $[T, M_\varphi]$ is the Marcinkiewicz space M_Ψ, where $\Psi(t) := t^{-1/n}\varphi(t)$.*

(c) *Suppose that φ_E is $(1/n')$-quasiconcave and $0 < \gamma_{\varphi_E} \le \delta_{\varphi_E} < 1/n'$. Then the largest r.i. space inside $[T, E]$ has fundamental function equivalent to $\Gamma(t) = t^{1/n}\varphi_E(t)$.*

For technical details on r.i. spaces E and their lower (resp. upper) dilation exponent γ_{φ_E} (resp. δ_{φ_E}) we refer to [6] and [70].

These results allow the formulation of an extended version of the classical Rellich-Kondrachov theorem on compactness of the Sobolev imbedding (for suitable $\Omega \subset \mathbb{R}^n$), which asserts that the imbedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \tag{2.6}$$

is compact for $1 \le q < np/(n-p)$ whenever $1 \le p < n$. In the case $q = np/(n-p)$, although Sobolev's theorem ensures boundedness, the imbedding is not compact. This can be interpreted in the following way: if we *fix* the range space to be some $L^q(\Omega)$ smaller than $L^{n'}(\Omega)$ (n' is the conjugate index of n), then the imbedding remains compact as long as the domain space $W_0^{1,p}(\Omega)$ does not reach the space $W_0^{1,nq/(n+q)}(\Omega)$ which is "too large" (i.e. the endpoint $nq/(n+q)$ is avoided). But, if we fix the range space to be some $L^q(\Omega)$ larger than $L^{n'}(\Omega)$, then the imbedding is compact for all domain spaces $W_0^{1,p}(\Omega)$ (since it is so for $W_0^{1,1}(\Omega)$) i.e. no endpoint occurs.

Setting $E(\Omega) := \{u\colon \Omega \to \mathbb{R} : u^* \in E\}$, and $\|u\|_{E(\Omega)} := \|u^*\|_E$ (which is a norm because E is r.i.), the Sobolev space $W_0^1E(\Omega)$ is defined as the closure of $C_0^1(\Omega)$ with respect to the norm $\|u\|_{W_0^1E(\Omega)} := \|u\|_{E(\Omega)} + \|\, |\nabla u|\, \|_{E(\Omega)}$ (see e.g. [15]). It follows that the inequality (2.4) is equivalent (by a generalized Poincaré inequality, [15, Lemma 4.2]) to boundedness of the inclusion

$$j\colon W_0^1F(\Omega) \hookrightarrow E(\Omega). \tag{2.7}$$

Hence, (2.7) is equivalent to boundedness of the kernel operator T in (2.5) from F to E. In view of the above results on optimal domains for the kernel operator T in (2.5), the *optimal r.i. Sobolev imbedding* is

$$j : W_0^1[T, E]^{ri}(\Omega) \hookrightarrow E(\Omega). \tag{2.8}$$

It turns out that compactness/noncompactness of the optimal r.i. Sobolev imbedding (2.8) is intimately connected to that of the associated kernel operator $T\colon [T, E]^{ri} \to E$. This is rather interesting, given that the extended operator $T\colon [T, E] \to X$ is *never* compact; see [20, Proposition 5.2(a)] and [21, Propositions 2.2(c) and 3.6(d)]. In this regard, we have the following result, [24, Theorems 3.7 and 3.9].

Theorem 2.11. *Let E be an r.i. space.*

(a) *If $t^{-1/n'}\varphi_E(t)$ is decreasing, then $T\colon [T,E]^{ri} \to E$ is not compact.*
(b) *$[T,E]^{ri} = L^1([0,1])$ and $T\colon [T,E]^{ri} \to E$ is compact if and only if $\lim_{t\to 0^+} t^{-1/n'}\varphi_E(t) = 0$.*

For $E = L^p([0,1])$, say, the condition (a) of Theorem 2.11 is satisfied whenever $p \geq n'$, so that $T\colon [T,E]^{ri} \to E$ is noncompact. Condition (b) is satisfied for $p < n'$, so that $T\colon [T,E]^{ri} \to E$ is compact in this case.

Theorem 2.11(a) can be "lifted" to obtain the following result, [24, Theorem 4.1].

Theorem 2.12. *Let E and F be r.i. spaces such that $F \subset [T,E]^{ri}$. If $T\colon F \to E$ is noncompact, then the Sobolev imbedding $j\colon W_0^1F(\Omega) \hookrightarrow E(\Omega)$ is bounded, but not compact.*

With the aid of this result, we have the extended version of the Rellich-Kondrachov theorem for the optimal r.i. Sobolev imbedding (2.8), [24, Theorems 4.3 and 4.4].

Theorem 2.13. *Let E be an r.i. space.*

(a) *If $t^{-1/n'}\varphi_E(t)$ is decreasing, then the optimal r.i. Sobolev imbedding* (2.8) *fails to be compact.*
(b) *If $\lim_{t\to 0^+} t^{-1/n'}\varphi_E(t) = 0$, then $[T,E]^{ri} = L^1([0,1])$ and we have compactness of the optimal r.i. Sobolev imbedding* (2.8)*.*

We end the current part concerning optimal Sobolev imbeddings by discussing the possibility of extending the previous results to the non-r.i. setting. For the operator T in (2.5) associated to Sobolev's inequality we know, by Theorem 2.9, that $T\colon [T,E]^{ri} \to E$ has a further genuine extension to $T\colon [T,E] \to E$ only in the case $E \neq L^{n',1}([0,1])$. Hence, if this is the case, then we may consider an *optimal Sobolev imbedding* more general than (2.8), namely

$$j : W_0^1[T,E](\Omega) \hookrightarrow E(\Omega). \tag{2.9}$$

However, difficulties arise in this attempt. Firstly, because $[T,E]$ will not be r.i., it is unclear how the spaces $[T,E](\Omega)$ and hence, also $W_0^1[T,E](\Omega)$, should even be defined. It turns out, due to specific properties of the kernel operator T and of the particular B.f.s. $[T,E]$, that the space $[T,E]$ is always a r.i. *quasi-Banach function space* and hence, that $W_0^1[T,E](\Omega)$ is always a quasi-Banach space (containing $W_0^1[T,X]^{ri}(\Omega)$), [25, Proposition 2.1].

Secondly, it is unclear whether the Sobolev imbedding (2.9) exists or not. The following result shows, at least by following this approach, that there is no possibility of extending the optimal result for r.i. Sobolev imbeddings to the non-r.i. case; [25, Theorems 1.1, 1.2 and 1.3].

Theorem 2.14. *Let E be a r.i. space.*

(a) *If $\lim_{t\to 0}\varphi_E(t)/t^{1/n'}=0$, then the optimal Sobolev imbedding* (2.9) *fails to exist for the space E.*

(b) *Let $E=\Lambda_\varphi$ be a Lorentz Λ-space such that $\varphi(t)/t^{1/n'}$ is equivalent to a decreasing function. Then the optimal Sobolev imbedding* (2.9) *exists for $E=\Lambda_\varphi$ but, it is not a further extension of the optimal r.i. Sobolev imbedding* (2.8).

(c) *Let E be a r.i. space whose Boyd indices satisfy*

$$0<\underline{\alpha}_X\le\overline{\alpha}_X<\frac{1}{n'}.$$

The optimal Sobolev imbedding (2.9) *exists for E but, it is not a further extension of the optimal r.i. Sobolev imbedding* (2.8).

Two of the most important operators acting in harmonic analysis are the Fourier transform and convolutions. Both are integral operators corresponding to $\mathbb{C}$-valued kernels and hence, the question of their optimal extension is again relevant. Let G be a compact abelian group with dual group Γ. Recall that $T\in\mathcal{L}(L^p(G))$, for $1\le p<\infty$, is a *Fourier p-multiplier operator* if it commutes with all translations, where $L^p(G)$ denotes the complex B.f.s. $L^p_{\mathbb{C}}(\lambda)$ with λ being Haar measure on G. Equivalently, there exists $\psi\in\ell^\infty(\Gamma)$ such that

$$\widehat{(Tf)}=\psi\widehat{f},\qquad f\in L^2\cap L^p(G), \tag{2.10}$$

where $\widehat{\cdot}$ denotes the Fourier transform, [72]. Since ψ is unique, T is typically denoted by T_ψ. Translations correspond to $\psi(\gamma)=\langle x,\gamma\rangle$ on Γ, for some $x\in G$, and convolutions to $\psi(\gamma)=\widehat{\mu}(\gamma)$ on Γ for some $\mu\in M(G)$, the space of all regular, $\mathbb{C}$-valued Borel measures on G. Here $\widehat{\mu}(\gamma):=\int_G\overline{\langle x,\gamma\rangle}\,d\mu(x)$, for $\gamma\in\Gamma$, is the *Fourier-Stieltjes transform* of μ. The (continuous) *convolution operator* $T_{\widehat{\mu}}$ acting in $L^p(G)$ is denoted by $C^{(p)}_\mu$ and is defined by $f\mapsto f*\mu$, for $f\in L^p(G)$, where $(f*\mu)(x):=\int_G f(x-y)\,d\mu(y)$ for λ-a.e. $x\in G$, belongs to $L^p(G)$ and satisfies $\|f*\mu\|_p\le|\mu|(G)\|f\|_p$. The vector measure $\nu_{T_{\widehat{\mu}}}$ corresponding to $T_{\widehat{\mu}}=C^{(p)}_\mu$ will be denoted more suggestively by $\nu^{(p)}_\mu$, that is, $\nu^{(p)}_\mu(A)=\mu*\chi_A$ for $A\in\mathcal{B}(G)$. The subclass corresponding to absolutely continuous measures $\mu\ll\lambda$ (ie. $\mu=\lambda_h$, for some $h\in L^1(G)$, where $\lambda_h(A):=\int_A h\,d\lambda$ on $\mathcal{B}(G)$) is quite different to that for general

$\mu \in M(G)$ and so we consider this first. We abbreviate $\nu_{\lambda_h}^{(p)}$ simply to $\nu_h^{(p)}$. For $\varphi \in L^{p'}(G) = L^p(G)^*$, where $p' := p/(p-1)$ is the conjugate index to p, it turns out that

$$\big(\varphi\nu_h^{(p)}\big)(A) := \langle \nu_h^{(p)}(A), \varphi\rangle = \int_A \varphi * \widetilde{h}\, d\lambda, \qquad A \in \mathcal{B}(G),$$

where $\widetilde{h}(x) := h(-x)$, for $x \in G$, is the *reflection* of h. The next result, [99, Lemma 2.2], collects together some basic properties of the vector measure $\nu_h^{(p)}$.

Theorem 2.15. *Let $1 \le p < \infty$ and fix $h \in L^1(G)$.*

(a) *The range of $\nu_h^{(p)}$ is a relatively compact subset of $L^p(G)$.*
(b) *Given any $A \in \mathcal{B}(G)$ its semivariation (cf. [32, p.2]) equals*

$$\|\nu_h^{(p)}\|(A) = \sup\left\{\int_A |\varphi * \widetilde{h}|\, d\lambda : \varphi \in L^{p'}(G),\ \|\varphi\|_{p'} \le 1\right\},$$

and satisfies

$$\|\widehat{h}\|_\infty\, \lambda(A) \le \|\nu_h^{(p)}\|(A) \le \|h\|_1\, (\lambda(A))^{1/p}.$$

(c) *If $h \ne 0$, then $\lambda \ll \nu_h^{(p)}$. Conversely, always $\nu_h^{(p)} \ll \lambda$.*

It follows from Theorem 2.15(c) that $C_h^{(p)}$ is λ-*determined* whenever $h \in L^1(G) \setminus \{0\}$. The following result, which is a combination of Theorem 1.1, Lemma 3.1 and Proposition 3.4 of [99], summarizes the essential properties of the o.c. optimal lattice domain space $L^1(\nu_h^{(p)})$ of $C_h^{(p)}$; see also [101, Proposition 7.46].

Theorem 2.16. *Let $1 \le p < \infty$ and fix $h \in L^1(G) \setminus \{0\}$.*

(a) *The inclusions*

$$L^p(G) \subseteq L^1(\nu_h^{(p)}) = L^1_w(\nu_h^{(p)}) \subseteq L^1(G)$$

hold and are continuous. Indeed,

$$\|f\|_{L^1(\nu_h^{(p)})} \le \|h\|_1\, \|f\|_p, \qquad f \in L^p(G),$$

and also

$$\|f\|_1 \le \|\widehat{h}\|_\infty^{-1}\, \|f\|_{L^1(\nu_h^{(p)})}, \qquad f \in L^1(\nu_h^{(p)}),$$

(b) $L^1(\nu_h^{(p)}) = \big\{f \in L^1(G) : \int_G |f|\cdot|\varphi * \widetilde{h}|\, d\lambda < \infty,\ \forall \varphi \in L^{p'}(G)\big\}$
and also
$L^1(\nu_h^{(p)}) = \big\{f \in L^1(G) : \big((\chi_A f) * h\big) \in L^p(G),\ \forall A \in \mathcal{B}(G)\big\}$.

Moreover, the norm of $f \in L^1(\nu_h^{(p)})$ *is given by*

$$\|f\|_{L^1(\nu_h^{(p)})} = \sup\left\{ \int_G |f| \cdot |\varphi * \widetilde{h}| \, d\lambda : \varphi \in L^{p'}(G), \ \|\varphi\|_{p'} \le 1 \right\}.$$

(c) $L^1(\nu_h^{(p)})$ *is a translation invariant subspace of* $L^1(G)$ *which is stable under formation of reflections and complex conjugates. Moreover, the extension* $I_{\nu_h^{(p)}} : L^1(\nu_h^{(p)}) \to L^p(G)$ *of* $C_h^{(p)}$ *to its o.c. optimal lattice domain* $L^1(\nu_h^{(p)})$ *is given by*

$$I_{\nu_h^{(p)}}(f) = h * f, \qquad f \in L^1(\nu_h^{(p)}). \tag{2.11}$$

It is known that $C_h^{(p)}$ is a compact operator in $L^p(G)$ for all $1 \le p < \infty$ and $h \in L^1(G)$. For $h \ne 0$, it turns out that the extended operator $I_{\nu_h^{(p)}}$ (see (2.11)) of $C_h^{(p)}$ to its o.c. optimal lattice domain $L^1(\nu_h^{(p)})$ is a compact operator iff $h \in L^p(G)$ iff the vector measure $\nu_h^{(p)} : \mathcal{B}(G) \to L^p(G)$ has finite variation iff $L^1(\nu_h^{(p)}) = L^1(G)$ is as large as possible; see [99, Theorem 1.2], where Theorem 1.9 above plays a crucial role in the proof. The following result, essentially Proposition 4.1 of [99], shows that more information about $I_{\nu_h^{(p)}}$ is available.

Theorem 2.17. *Let* $1 \le p < \infty$ *and* $h \in L^1(G) \setminus \{0\}$.

(a) *If* $h \notin L^p(G)$, *then* $I_{\nu_h^{(p)}} : L^1(\nu_h^{(p)}) \to L^p(G)$ *is not compact and both the following inclusions are proper:*

$$L^p(G) \subseteq L^1(\nu_h^{(p)}) \subseteq L^1(G). \tag{2.12}$$

The first inclusion in (2.12) *is proper for every* $h \in L^1(G) \setminus \{0\}$. *In particular, the o.c. optimal lattice domain* $L^1(\nu_h^{(p)})$ *of* $C_h^{(p)}$ *is always genuinely larger than* $L^p(G)$.

(b) *There exists* $h \in L^1(G)$ *with* $\bigcup_{1<p<\infty} L^1(\nu_h^{(p)}) \subsetneqq L^1(G)$.

For further results we refer to [99] and [101, Ch.7, §7.3].

We now turn our attention to $C_\mu^{(p)}$ with $\mu \in M(G) \setminus L^1(G)$. Of relevance are the measures in $M_0(G) := \{\mu \in M(G) : \widehat{\mu} \in c_0(\Gamma)\}$. Indeed, it turns out for $1 < p < \infty$ and $\mu \in M(G)$ that $C_\mu^{(p)}$ is compact in $L^p(G)$ iff $\mu \in M_0(G)$ iff the range of the vector measure $\nu_\mu^{(p)}$ is a relatively compact subset of $L^p(G)$, [100, Proposition 2.3]. For arbitrary $\mu \in M(G)$ it is the case that $\nu_\mu^{(p)} \ll \lambda$ and, if $\mu \ne 0$, then also, $\lambda \ll \nu_\mu^{(p)}$, [100, Proposition 2.4]. In particular, $C_\mu^{(p)}$ is λ-*determined* whenever $\mu \in M(G) \setminus \{0\}$. Moreover, the statement of Theorem 2.16 remains valid (throughout) if we replace $h \simeq \lambda_h$ (resp.

$\nu_h^{(p)}$) with $\mu \in M(G)$ (resp. $\nu_\mu^{(p)}$); see [100, Theorem 1.1 & Corollary 3.2] and [101, Proposition 7.60]. Compactness of the operators $C_\mu^{(p)}$ was characterized above (eg. in terms of $M_0(G)$, say). For their optimal extension $I_{\nu_\mu^{(p)}}$ we have the following result, which is a combination of Theorem 1.2, Remark 4.2(b) and Proposition 4.3 of [100].

Theorem 2.18. *Let $1 \leq p < \infty$ and fix $\mu \in M(G) \setminus \{0\}$. Then the following assertions are equivalent.*

(a) *The extension $I_{\nu_\mu^{(p)}} : L^1(\nu_\mu^{(p)}) \to L^p(G)$ of $C_\mu^{(p)}$ to its o.c. optimal lattice domain $L^1(\nu_\mu^{(p)})$ is a compact operator.*
(b) *$\mu = \lambda_h$ for some $h \in L^p(G)$.*
(c) *The vector measure $\nu_\mu^{(p)} : \mathcal{B}(G) \to L^p(G)$ has finite variation.*
(d) *$L^1(\nu_\mu^{(p)}) = L^1(G)$.*

We now take a closer look at the spaces $L^1(\nu_\mu^{(p)})$ for arbitrary $\mu \in M(G)$; phenomena quite different to the case of $\mu \ll \lambda$ can occur (which is covered by Theorem 2.17 above). For $a \in G$, we denote the Dirac point mass at a by δ_a.

Theorem 2.19. *Let $1 < p < \infty$ and $\mu \in M(G)$.*

(a) *If $\operatorname{supp}(\mu) \neq \mathrm{G}$ and $a \notin \operatorname{supp}(\lambda)$, then $L^1(\nu_{\mu+\delta_a}^{(p)}) = L^p(G)$.*
(b) *If $a \in G$ and $\mu \in M_0(G)$, then $L^1(\nu_{\mu+\delta_a}^{(p)}) = L^p(G)$.*
(c) *If there exists $\eta \in M(G)$ satisfying $\mu * \eta = \delta_0$, then $L^1(\nu_\mu^{(p)}) = L^p(G)$.*
(d) *If $\mu \geq 0$ and $L^1(\nu_\mu^{(p)}) \neq L^p(G)$ for some $1 < p < \infty$, then μ is a continuous measure (i.e. $\mu(\{a\}) = 0$ for all $a \in G$).*
(e) *The inclusion $L^p(G) \subseteq L^1(\nu_\mu^{(p)})$ is proper if $\mu \in M_0(G) \setminus \{0\}$.*

For part (a) we refer to [101, Remark 7.75], for (b) to [101, Proposition 7.77], for (c) to [101, Corollary 7.79] and for (d) to [101, Corollary 7.76]. Part (e) is [100, Proposition 4.5]. Cases (a)-(c) in Theorem 2.19 show that $C_\mu^{(p)}$ may already be defined on its o.c. optimal lattice domain and no further extension is possible. Case (e) in Theorem 2.19 exhibits a large class of measures μ where the optimal extension is always *genuine*. Theorem 2.18 characterizes those μ for which the optimal extension is to the largest possible domain space, namely $L^1(G)$.

Consider now $G := \mathbb{T}$, the circle group, in which case $\Gamma = \mathbb{Z}$. For $1 \leq p \leq 2$, it is known that the Fourier transform $F_p : L^p(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$ in injective and that it is continuous, because of the *Hausdorff-Young* *inequality*

$$\|\widehat{f}\|_{p'} \leq \|f\|_p, \qquad f \in L^p(\mathbb{T}).$$

Of course, $F_p f := \widehat{f}$ for each $f \in L^p(\mathbb{T}) \subseteq L^1(\mathbb{T})$. As for Sobolev's inequality, discussed in Section 2, one may ask whether the Hausdorff-Young inequality is optimal. Now, for $X := L^p(\mathbb{T})$ and $E := \ell^{p'}(\mathbb{Z})$ and $T := F_p$, we have $\nu_{F_p}(A) = F_p(\chi_A) = \widehat{\chi_A}$ for $A \in \mathcal{B}(\mathbb{T})$. Note that the B.f.s. X has o.c. norm and that the codomain space E is reflexive. Accordingly, ν_{F_p} is σ-additive and $L^1(\nu_{F_p}) = L^1_w(\nu_{F_p})$. It is known that the vector measure $\nu_{F_p} : \mathcal{B}(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$ is mutually absolutely continuous with respect to Haar measure λ on $\mathbb{T}$ (hence, F_p is a λ-*determined* operator), that it has infinite variation for $1 < p \leq 2$ (with ν_{F_1} having finite variation), and that the range of ν_{F_p}, for $1 \leq p < \infty$, is not a relatively compact subset of $\ell^{p'}(\mathbb{Z})$; see Lemma 2.1, Remark 2.2 and Corollary 2.5 of [87]. Moreover, $L^1(\nu_{F_p}) \subseteq L^1(\mathbb{T})$ with $\|f\|_1 \leq \|f\|_{L^1(\nu_{F_p})}$, for each $f \in L^1(\nu_{F_p})$, and the extension $I_{\nu_{F_p}} : L^1(\nu_{F_p}) \to \ell^{p'}(\mathbb{Z})$ of F_p is again the map $f \mapsto \widehat{f}$, for $f \in L^1(\nu_{F_p})$, [87, Theorem 1.1(iii)]. For simplicity, we adopt the notation of [87] and, henceforth, write $\mathbf{F}^p(\mathbb{T}) := L^1(\nu_{F_p})$.

We now turn to a more concrete description of $\mathbf{F}^p(\mathbb{T})$. First,

$$\|f\|_{\mathbf{F}^p(\mathbb{T})} = \sup\left\{ \int_{\mathbb{T}} |f| \cdot |\check{\phi}| \, d\lambda : \phi \in \ell^p(\mathbb{Z}),\ \|\phi\|_p \leq 1 \right\},$$

where $\check{\phi}$ is the inverse Fourier transform of $\phi \in \ell^p(\mathbb{Z}) = \ell^{p'}(\mathbb{Z})^* \subseteq \ell^2(\mathbb{Z})$. Given $1 \leq p \leq 2$, define a vector subspace $V^p(\mathbb{T})$ of $L^{p'}(\mathbb{T})$ by

$$V^p(\mathbb{T}) := \{h \in L^{p'}(\mathbb{T}) : h = \check{\varphi} \ \text{ for some } \ \varphi \in \ell^p(\mathbb{Z})\}.$$

For $p = 2$, Plancherel's theorem implies that $V^2(\mathbb{T}) = L^2(\mathbb{T})$. It is known that there exists $f \in C(\mathbb{T})$ such that $\widehat{f} \notin \ell^r(\mathbb{Z})$ for all $1 \leq r < 2$ and so the containment $V^p(\mathbb{T}) \subseteq L^{p'}(\mathbb{T})$ is *proper* for $1 \leq p < 2$. For each $f \in L^1(\mathbb{T})$, define a linear map $S_f : L^\infty(\mathbb{T}) \to c_0(\mathbb{Z})$ by $S_f : g \mapsto \widehat{gf}$ for $g \in L^\infty(\mathbb{T})$. Clearly $\|S_f\| \leq \|f\|_1$. For each $1 \leq p \leq 2$ and each continuous operator $R : L^\infty(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$, let $\|R\|_{\infty,p'} := \sup_{\|g\|_\infty \leq 1} \|Rg\|_{p'}$ denote its operator norm. If $f \in L^1(\mathbb{T})$ has the property that the range $S_f(L^\infty(\mathbb{T})) \subseteq \ell^{p'}(\mathbb{Z})$, then the Closed Graph Theorem implies that $\|S_f\|_{\infty,p'} < \infty$. For the next result, see [87, Theorem 1.2].

Theorem 2.20. *Let $1 \le p \le 2$. Each of the spaces*

$$\Delta^p(\mathbb{T}) = \left\{ f \in L^1(\mathbb{T}) : \int_{\mathbb{T}} |fg| \, d\lambda < \infty, \ \forall g \in V^p(\mathbb{T}) \right\},$$

$$\Phi^p(\mathbb{T}) = \left\{ f \in L^1(\mathbb{T}) : \widehat{f\chi_{_A}} \in \ell^{p'}(\mathbb{Z}), \ \forall A \in \mathcal{B}(\mathbb{T}) \right\},$$

$$\Gamma^p(\mathbb{T}) = \left\{ f \in L^1(\mathbb{T}) : S_f(L^\infty(\mathbb{T})) \subseteq \ell^{p'}(\mathbb{Z}) \right\}, \qquad (2.13)$$

coincides with the o.c. optimal lattice domain $\mathbf{F}^p(\mathbb{T})$ of the Hausdorff-Young inequality. Moreover, in the case of (2.13)*, the operator norm $\|S_f\|_{\infty,p'}$ is equivalent to $\|f\|_{\mathbf{F}^p(\mathbb{T})}$, for $f \in \mathbf{F}^p(\mathbb{T})$.*

For $p = 1, 2$ it turns out that $\mathbf{F}^1(\mathbb{T}) = L^1(\mathbb{T})$ and $\mathbf{F}^2(\mathbb{T}) = L^2(\mathbb{T})$. So, both maps $F_1 : L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z})$ and $F_2 : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ are already defined on their o.c. optimal lattice domain; no further extension is possible. Is $\mathbf{F}^p(\mathbb{T})$ genuinely larger than $L^p(\mathbb{T})$ for $1 < p < 2$? The answer is given by the following result, [87, Theorem 1.4].

Theorem 2.21. *For $1 < p < 2$, the following inclusions are proper:*

$$L^p(\mathbb{T}) \subseteq \mathbf{F}^p(\mathbb{T}) \subseteq L^1(\mathbb{T}).$$

It is also shown in [87] that, for each $1 < r < p$, the space $\mathbf{F}^p(\mathbb{T})$ is not contained in $L^r(\mathbb{T})$ and, that $L^r(\mathbb{T})$ is not contained in $\mathbf{F}^p(\mathbb{T})$ for any $1 \le r < p$. It is also established in [87] that $\mathbf{F}^p(\mathbb{T})$ is a translation invariant subspace of $L^1(\mathbb{T})$ and that it is a weakly sequentially complete B.f.s. with the σ-Fatou property. Moreover, the translation operators $\tau_w f(z) = f(zw^{-1})$, for $w, z \in \mathbb{T}$, are continuous in $\mathbf{F}^p(\mathbb{T})$ and τ_w converges to the identity operator (for the strong operator topology) as $w \mapsto 1$. Accordingly, $\mathbf{F}^p(\mathbb{T})$ is a *homogeneous* Banach space and so is well suited for harmonic analysis.

3. Aspects of spectral integration

A rich source of vector measures arises in spectral theory. For instance, the resolution of the identity of a normal operator in a Hilbert space is a σ-additive projection-valued measure (i.e. *spectral measure*). Such vector (= operator-valued) measures were extended to the Banach space setting by N. Dunford, [45]. The theory of spectral measures and integration in Banach and Fréchet spaces is somewhat more complicated than in Hilbert spaces (where it is more transparent because of the Mackey-Wermer theorem, [42, Proposition 8.2]). Nevertheless, there have been significant advances in this theory over the past 15 years or so. We begin with developments that occured in the general theory; the latter half of the section is devoted to applications.

Let X be a Fréchet space and $\mathcal{L}(X)$ denote the space of all continuous linear operators of X into itself. The identity operator in X is denoted by I. The strong operator topology τ_s (briefly, sot) in $\mathcal{L}(X)$ is generated by the seminorms

$$q_{x,n} : T \mapsto \|Tx\|_n, \qquad T \in \mathcal{L}(X), \tag{3.1}$$

where $x \in X$ is arbitrary and $\{\| \cdot \|_n\}_{n=1}^{\infty}$ is a sequence of continuous seminorms determining the topology of X. Then $E = \mathcal{L}_s(X)$ denotes the quasicomplete lcHs $\mathcal{L}(X)$ equipped with the continuous seminorms $Q := \{q_{x,n} : x \in X, n \in \mathbb{N}\}$ as given by (3.1).

A collection $\mathcal{M} \subseteq \mathcal{L}(X)$ of commuting projections is a *Boolean algebra* (briefly, B.a.) if $0, I \in \mathcal{M}$ and $\mathcal{M}$ is a B.a. for the partial order $\leq$ defined by $P \leq R$ iff $PR = P = RP$ (equivalently, their ranges satisfy $PX \subseteq RX$). The B.a. operations $\vee, \wedge$ and complementation in $\mathcal{M}$ are then given by $P \wedge R = PR$ and $P \vee R = P + R - PR$ and $P^c = I - P$. If $\mathcal{M}$ is equicontinuous in $\mathcal{L}(X)$, then $\mathcal{M}$ is called *equicontinuous* (or *bounded* if X is Banach). We will discuss only such B.a.'s of projections, although important examples occur in classical analysis which are not equicontinuous; see [84; 111] and [117, Ch.III Example 18]. It is remarkable that every B.a. of projections which is merely σ-complete as an abstract B.a. is already equicontinuous; see [5, Theorem 2.2] for X a Banach space, [134, Proposition 1.2] for X a Fréchet space, and [9] for some other classes of lcHs' X. Here *abstractly σ-complete* means every countable subset $A \subseteq \mathcal{M}$ has a greatest lower bound, denoted by $\wedge A$ (equivalently a least upper bound, denoted by $\vee A$).

According to a result of M.H. Stone, every B.a. of projections $\mathcal{M} \subseteq \mathcal{L}(X)$ is isomorphic to the algebra of all closed-open sets $Co(\Omega_{\mathcal{M}})$ of some totally disconnected, compact Hausdorff space $\Omega_{\mathcal{M}}$. This isomorphism is finitely additive as a set function from $Co(\Omega_{\mathcal{M}})$ into $\mathcal{L}(X)$. If $\mathcal{M}$ is abstractly σ-complete, then $\Omega_{\mathcal{M}}$ is basically disconnected and if $\mathcal{M}$ is abstractly complete (i.e. every $A \subseteq \mathcal{M}$ has a greatest lower bound $\wedge A$; equivalently, a least upper bound $\vee A$), then $\Omega_{\mathcal{M}}$ is extremely disconnected; see eg. [117, Ch.II] for a discussion of these classical facts. In particular, every B.a. of projections can be represented as the range $P(\Sigma) := \{P(A) : A \in \Sigma\}$ of some finitely additive *spectral measure* $P : \Sigma \to \mathcal{L}(X)$ and vice versa (with Σ an algebra of subsets of some set $\Omega \neq \emptyset$), where P satisfies $P(\emptyset) = 0$ and $P(\Omega) = I$ and P is multiplicative (i.e. $P(A \cap B) = P(A)P(B)$ for $A, B \in \Sigma$).

A B.a. of projections $\mathcal{M} \subseteq \mathcal{L}(X)$ is *Bade complete* (resp. *Bade σ-complete*) if it is abstractly complete (resp. abstractly σ-complete) and

$$(\wedge_\alpha B_\alpha)X = \cap_\alpha (B_\alpha X) \;\text{ and }\; (\vee_\alpha B_\alpha)X = \overline{\mathrm{sp}\{\cup_\alpha (B_\alpha X)\}}, \tag{3.2}$$

whenever $\{B_\alpha\}$ is a family (resp. countable family) of elements from $\mathcal{M}$, [5; 134]. The interaction between the order properties of $\mathcal{M}$ and the topology of X (via (3.2)) has some far reaching consequences.

Theorem 3.1. *Let X be a Fréchet space and $\mathcal{M} \subseteq \mathcal{L}(X)$ be a B.a. of projections.*

(a) *If $\mathcal{M}$ is Bade complete, then it is a closed (hence, also complete) subset of $\mathcal{L}_s(X)$.*
(b) *If $\mathcal{M}$ is Bade σ-complete, then its closure $\overline{\mathcal{M}}_s$ in $\mathcal{L}_s(X)$ is a Bade complete B.a. of projections.*
(c) *If $\mathcal{M}$ is Bade σ-complete (resp. Bade complete), then $\mathcal{M}$ is the range of a σ-additive $\mathcal{L}_s(X)$-valued spectral measure defined on the Baire (resp. Borel) subsets of the Stone space $\Omega_{\mathcal{M}}$ of $\mathcal{M}$.*

For Banach spaces, (a) and (b) occur in [5] and for Fréchet spaces in [40, §4]. Part (c) is folklore; for a discussion and proof see [92, §4] and [93], for example, and the references therein.

If X is separable, then every Bade σ-complete B.a. of projections $\mathcal{M}$ is Bade complete; see [5, p.350] and [40, Proposition 4.3]. If $\mathcal{M}$ possesses a *cyclic vector* (i.e. $X = \overline{\mathrm{sp}\{Bx : B \in \mathcal{M}\}}$ for some $x \in X$), then again $\mathcal{M}$ is Bade complete; the same is true if $\mathcal{M}$ is *countably decomposable* (i.e. every pairwise disjoint family of elements from $\mathcal{M}$ is at most countable). For these claims and further sufficient conditions on $\mathcal{M}$ see [94]. If we restrict X to the class of Banach spaces, even more is known. For instance, X has the property that every bounded B.a. of projections $\mathcal{M} \subseteq \mathcal{L}_s(X)$ which is τ_s-closed is Bade complete iff X does not contain an isomorphic copy of c_0, [64]. Or, if X is separable, then a B.a. of projections in X is abstractly σ-complete iff it is abstractly complete, [116, Corollary 2.1]; this is a consequence of the fact that in any weakly compactly generated Banach space (separable spaces have this property) the τ_s-closure of any abstractly σ-complete B.a. of projections is Bade complete, [116, Theorem 2]. Recently it was shown that a Banach space X has the property that the τ_s-closure of every abstractly σ-complete B.a. of projections in X is Bade complete iff X does not contain an isomorphic copy of ℓ^∞, [35]. This is of interest because an abstractly σ-complete B.a. of projections in a Banach space not containing a copy of ℓ^∞ need not be Bade complete or even Bade σ-complete, [116, Remark 2].

In view of Theorem 3.1(b), it might be anticipated that every Bade σ-complete B.a. of projections in a Banach space X is at least a sequentially closed subset of $\mathcal{L}_s(X)$. For purely atomic B.a.'s this was known to be the case, [115], but, recently this question was answered in the *negative* in [58], even for Hilbert spaces! The paper [58] is also of interest because it exhibits a large class of *non-atomic*, Bade σ-complete (but, not Bade complete) B.a.'s of projections in the *non-separable* Banach space $ca(\Sigma)$ consisting of all σ-additive, $\mathbb{R}$-valued measures on a measurable space (Ω, Σ) equipped with the total variation norm. Such types of examples were missing in the past.

Non-trivial, concrete examples of Bade complete and σ-complete B.a.'s of projections in *Fréchet* spaces X have been somewhat scarce in the past; see [89; 123; 134; 135], for some such examples. In recent years this list of examples has been significantly extended and reveals (in certain cases) an intimate connection between properties of $\mathcal{M}$ (eg. Dunford's boundedness criterion, boundedly σ-complete, finite τ_s-variation) and geometric properties of X (eg. nuclear, Montel, Radon-Nikodým property); see [10; 11; 12; 13; 90; 95].

There is also a converse to Theorem 3.1.

Theorem 3.2. *Let X be a Fréchet space and $P : \Sigma \to \mathcal{L}_s(X)$ be a σ-additive spectral measure defined on a σ-algebra Σ.*

(a) *The range $P(\Sigma) \subseteq \mathcal{L}(X)$ is an equicontinuous, Bade σ-complete B.a. of projections.*
(b) *The B.a. of projections $P(\Sigma)$ is Bade complete if and only if $P(\Sigma)$ is a closed subset of $\mathcal{L}_s(X)$.*

For the proof and a discussion of this result, together with its historical origins (for X normable and non-normable), see [92, Section 3] and [93, Theorem 2, Remark 4.3] and the references therein.

Of particular interest is the case when Σ equals the Borel subsets $\mathcal{B}(K)$ of some compact set $K \subseteq \mathbb{C}$ (or $K \subseteq \mathbb{C} \cup \{\infty\}$ if X is non-normable). It is assumed that K is minimal, that is, it equals the *support* of the spectral measure P (in the sense of [42, p.122]) and that the identity function on K belongs to $L^1(P)$, in which case the operator $I_P(z) = \int_K z \, dP(z)$ is called a *scalar-type spectral operator* and P is its (unique) *resolution of the identity.* For X a Banach space, such operators have been extensively studied in [42; 45; 117]. It turns out, for X a separable Banach space, that *every* Bade σ-complete (= Bade complete) B.a. of projections $\mathcal{M} \subseteq \mathcal{L}(X)$ coincides with the resolution of the identity of some scalar-type spectral operator [103, Proposition 2]. The proof of this result depends on the existence of *Bade functionals*: i.e. given any $x \in X$ there exists $x^* \in X^*$ with the properties that $\langle Rx, x^* \rangle \geq 0$ for all $R \in \mathcal{M}$ and $Bx = 0$ whenever $B \in \mathcal{M}$ satisfies $\langle Bx, x^* \rangle = 0$, [5, Theorem 3.1]. Unfortunately, this remarkable fact fails to hold in a general Fréchet space X. Indeed, a Fréchet space X has the property that *every* Bade σ-complete B.a.

of projections $\mathcal{M} \subseteq \mathcal{L}(X)$ possesses Bade functionals (for arbitrary $x \in X$) iff X does not contain an isomorphic copy of the Fréchet sequence space $\mathbb{C}^{\mathbb{N}}$, [114, Theorem 2]. Nevertheless, it remains true that every Bade σ-complete (= Bade complete) B.a. of projections in a separable Fréchet space coincides with the resolution of the identity of some scalar-type spectral operator, [118].

For X a Fréchet space and $P : \Sigma \to \mathcal{L}_s(X)$ a σ-additive spectral measure, the integration operator $I_P : L^1(P) \to \mathcal{L}_s(X)$ is of central importance. Indeed, with respect to pointwise multiplication the lc-Riesz space $L^1(P)$ is also a commutative lc-algebra with identity and is topologically τ_s-complete iff $P(\Sigma)$ is a closed subset of $\mathcal{L}_s(X)$, [40, Proposition 1.4]. In the case when $P(\Sigma)$ is τ_s-closed, the integration operator I_P is an isomorphism of the complete lc-algebra $L^1(P)$ onto the closed operator algebra in $\mathcal{L}_s(X)$ generated by $P(\Sigma)$, [40, Proposition 1.5]. It should be pointed out that $L^1(P) = L^\infty(P)$, as vector spaces, whenever X is a Banach space, [117, Proposition V.4]. For each $x \in X$, let $P(\Sigma)[x]$ denote the closed subspace of X generated by $\{P(A)x : A \in \Sigma\}$; it is called the *cyclic space* generated by x. The X-valued vector measure defined in Σ by $A \mapsto P(A)x$ is denoted by Px. An important fact is that $L^1(P) = \bigcap_{x\in X} L^1(Px)$, [95, Lemma 2.2]. The following result, [41, Proposition 2.1], a converse to Theorem 1.3, reveals the intimate connection between B.a.'s of projections, spectral measures and certain aspects from the theory of order and positivity that arise via Banach lattices (and more general Riesz spaces); see also [39]. For the terminology of undefined notions we refer to [3].

Theorem 3.3. *Let X be a Fréchet space and $\mathcal{M} \subseteq \mathcal{L}(X)$ be a Bade complete B.a. of projections, displayed as the range of some spectral measure $P : \Sigma \to \mathcal{L}_s(X)$, that is, $\mathcal{M} = P(\Sigma)$. Then, for each $x \in X$, the integration operator $I_{Px} : L^1(Px) \to \mathcal{M}[x]$ induces on the cyclic space $\mathcal{M}[x]$ the structure of a Dedekind complete Fréchet lattice with Lebesgue topology in which x is a weak order unit. The absolute value of an element $\int_\Omega f\ dPx \in \mathcal{M}[x]$ is the element $\int_\Omega |f|\ dPx \in \mathcal{M}[x]$. Moreover, I_{Px} is a Riesz and topological isomorphism and the absolute value mapping on $\mathcal{M}[x]$ is continuous.*

Concerning Theorem 3.3, if there exists a *cyclic vector* $x_0 \in X$ for $\mathcal{M}$, then the integration operator induces on X the structure of a Dedekind complete Fréchet lattice (= Banach lattice if X is normable) with a Lebesgue topology (= order continuous norm if X is normable) which is equivalent to the original topology in X and such that x_0 is a weak order unit. Moreover, with respect to this Fréchet lattice topology, I_P is a *positive operator* and the B.a. $\mathcal{M}$ may be identified with the B.a. of all *band projections*, [41, Proposition 2.1]. For X a Banach space, this fact goes essentially back to A.I. Veksler, [133]. For X non-normable, to describe $L^1(P)$ "concretely" is, in general, rather difficult. Some illuminating and non-trivial examples occur in [10; 11; 12], where $L^1(P)$ is identified with a certain space of multiplication operators acting on X.

To decide whether a *particular* operator acting in a given Banach space is actually scalar-type spectral (briefly, scalar) can be rather difficult; see [45], for example. Here we only make some relevant comments in the direction of harmonic analysis. That certain translation and convolution operators in L^p-spaces over $\mathbb{Z}$, for $p \neq 2$, fail to be scalar operators goes back to U. Fixman, [57], and G.L. Krabbe, [69]. Translation operators in $L^p(G)$, with G an arbitrary locally compact abelian group and $p \neq 2$, were shown by T.A. Gillespie to be scalar operators iff they have finite spectrum, [63]. If one is prepared to relax the topology in $L^p(G)$, then it may happen that non-scalar translation operators in $L^p(G)$ *do* become scalar operators when extended to act in a superspace $X_p(G)$ (not necessarily Fréchet) which contains $L^p(G)$ continuously, [60; 61], but not always. As noted in Section 2, translations and convolutions are special cases of Fourier multiplier operators. For $G = \mathbb{T}$ (or any compact metrizable abelian group), those Fourier p-multiplier operators $T_\psi \in \mathcal{L}(L^p(G))$ which are scalar are characterized in [85]; see also [86]. Namely, for each λ in the countable set $\psi(\Gamma) \subseteq \mathbb{C}$ the idempotent $\chi_{\psi^{-1}(\{\lambda\})}$ should be a p-multiplier for G and the pairwise disjoint family of Fourier p-multiplier projections $\{T\chi_{\psi^{-1}(\{\lambda\})} : \lambda \in \psi(\Gamma)\} \subseteq \mathcal{L}(L^p(G))$ should be a *Littlewood-Paley p-decomposition* for G. For $G = \mathbb{R}^N$ and $\psi : \mathbb{R}^N \to \mathbb{C}$ a polynomial, (2.10) can be used to define a closed, densely defined, unbounded Fourier p-multiplier operator which is, of course, none-other than a constant coefficient linear partial differential operator in $L^p(\mathbb{R}^N)$. Such operators are rarely (unbounded) scalar operators for $p \neq 2$, [1]; see also [2; 113] for results concerning the spectrality of *matrix-valued* Fourier p-multiplier and differential operators in L^p-spaces.

We also mention some recent directions where vector and operator-valued measures occur in vector-valued harmonic analysis. Let G be a lca group and $\lambda : \mathcal{B}(G) \to [0,\infty]$ denote Haar measure. Given $1 \leq p < \infty$ and a Hilbert space $\mathcal{H}$, let $L^p(G,\mathcal{H})$ be the Banach space of (equivalence classes of) strongly λ-measurable functions $f : G \to \mathcal{H}$ such that the norm $\|f\|_p := \Big(\int_G \|f(u)\|_{\mathcal{H}}^p \, d\lambda(u)\Big)^{1/p} < \infty$. The Fourier transform of a function $f \in L^1(G, \mathcal{H})$ is defined in the natural way:

$$\widehat{f}(\gamma) := \int_G \overline{\langle u, \gamma \rangle} f(u) \, d\lambda(u), \qquad \gamma \in \Gamma,$$

the integral being a $\mathcal{H}$-valued Bochner integral. An operator $T \in \mathcal{L}(L^p(G, \mathcal{H}))$ is called a *Fourier p-multiplier operator* if there exists a measurable function $\Phi : \Gamma \to \mathcal{L}(\mathcal{H})$ such that Φ is essentially bounded in the operator norm of $\mathcal{L}(\mathcal{H})$ and for each $f \in L^1 \cap L^2 \cap L^p(G, \mathcal{H})$ the equality $(\widehat{Tf})(\gamma) = \Phi(\gamma)\widehat{f}(\gamma)$ holds a.e. on Γ (compare with (2.10)). Here the measurability of Φ means that the $\mathcal{H}$-valued function $\gamma \mapsto \Phi(\gamma)h$ is strongly measurable for each $h \in \mathcal{H}$. As for scalar-valued harmonic analysis, it turns out that T is a Fourier p-multiplier operator iff it commutes with each translation operator in $\mathcal{L}(L^p(G, \mathcal{H}))$, [62, Proposition 2.8]. The monograph [62] is mainly concerned with an analysis of the more special case when

$$\Phi(\gamma) = \widehat{\mu}(\gamma) := \int_G \overline{\langle u, \gamma \rangle} \, d\mu(u), \qquad \gamma \in \Gamma,$$

is the Fourier-Stieltjes transform $\widehat{\mu} : \Gamma \to \mathcal{L}(\mathcal{H})$ of a regular *operator-valued measure* $\mu : \mathcal{B}(G) \to \mathcal{L}_s(\mathcal{H})$. A collection of negative results illustrates decisively how known L^p multiplier results in the scalar setting can break down in the vector-valued setting when $p \neq 2$. For instance, because the operator-valued measure μ is not, generally, selfadjoint-valued, it can happen that $\widehat{\mu} : \Gamma \to \mathcal{L}(\mathcal{H})$ is a Fourier p-multiplier but *not* a Fourier p'-multiplier, where $\frac{1}{p} + \frac{1}{p'} = 1$. Chapter 4 of [62] is devoted to the case when $\mu : \mathcal{B}(G) \to \mathcal{L}_s(\mathcal{H})$ is a *spectral measure*. It is shown that whenever $(G, \mathcal{B}(G), \lambda)$ is a separable measure space (with G infinite), $\mathcal{H} = L^2(G)$ and μ is the canonical (selfadjoint) spectral measure acting in $L^2(G)$ via multiplication with χ_A, for $A \in \mathcal{B}(G)$, then $T_{\widehat{\mu}}$ is a Fourier p-multiplier operator iff $p = 2$, [62, Theorem 4.7]. As a consequence: according to Stone's theorem, the translation group $\Phi : \Gamma \to \mathcal{L}_s(L^2(\Gamma))$, which is bounded and τ_s-continuous, has the form $\Phi = \widehat{P}$ for some regular, selfadjoint spectral measure $P : \mathcal{B}(G) \to \mathcal{L}_s(L^2(G))$. By the previous mentioned fact it follows that $T_{\widehat{P}}$ is a Fourier p-multiplier operator (acting in $L^p(G, L^2(G))$) iff $p = 2$. Of course, [62] has further results than just the sample alluded to above; see also [131] and the references therein.

A fundamental question is to determine criteria which ensure that the sum and product of two commuting scalar operators acting in a Banach space are again scalar: an example of C.A. McCarthy shows, even in a separable reflexive Banach space, that this is not always the case, [83]. Since commutativity of the scalar operators is equivalent to the commutativity of their resolutions of the identity (a consequence of [45, XV Corollary 3.7]), the above question is intimately related to the problem of determining criteria which ensure that the B.a. generated by two commuting, bounded B.a.'s of projections is again bounded. An account of what was known up to 1970 in regard to these questions can be found in [45, pp.2098-2101]. Much research concentrated on identifying classes of Banach spaces in which the answer is positive. For instance, C.A. McCarthy established that all L^p-spaces, for $1 \leq p < \infty$, have this property and (together with W. Littman and N. Riviére) also their complemented subspaces; see [45, pp.2099-2100], for example. This is also the case for all Grothendieck spaces with the Dunford-Pettis property, [110], and the class of all hereditarily indecomposable Banach spaces, [112]. Also [43] is relevant to these questions. However, not so many results have appeared in this regard, since it is difficult to identify geometric conditions on a Banach space X which ensure that *all* pairs of commuting, bounded B.a.'s of projections in X automatically have a uniformly bounded product B.a. Perhaps the most recent significant result in this direction is the following one, due to T.A. Gillespie, [65].

Theorem 3.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be commuting, bounded B.a.'s of projections in a Banach space X. Then the B.a. of projections $\mathcal{M} \vee \mathcal{N}$ generated by $\mathcal{M}$ and $\mathcal{N}$ is also bounded, in each of the following cases.*

(a) *X is a Banach lattice.*
(b) *X is a closed subspace of any p-concave Banach lattice (p finite).*
(c) *X is a complemented subspace of any $\mathcal{L}^\infty$-space.*
(d) *X is a closed subspace of $\mathcal{L}^p$ for any $1 \leq p < \infty$.*
(e) *X has local unconditional structure (briefly, l.u.st.).*

In the recent article [104], the viewpoint was taken that the geometry of X is not the only relevant ingredient; an important property of the individual B.a.'s concerned (when available) can also play a crucial role. This is the notion of R-boundedness, introduced by E. Berkson and T.A. Gillespie in [7] (where it is called the R-property), but already explicit in earlier work of J. Bourgain, [14]. For a Banach space X, a non-empty collection $\mathcal{T} \subseteq \mathcal{L}(X)$ is called *R-bounded* if there exists a constant $M \geq 0$ such that

$$\Big(\int_0^1 \Big\| \sum_{j=1}^n r_j(t) T_j x_j \Big\|^2 \, dt\Big)^{1/2} \leq M \Big(\int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|^2 \, dt\Big)^{1/2}$$

for all $T_1, \ldots, T_n \in \mathcal{T}$, all $x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$, where $\{r_j\}_{j=1}^{\infty}$ is the sequence of Rademacher functions on $[0,1]$. Clearly every R-bounded collection is uniformly bounded in $\mathcal{L}(X)$. An important fact is the following one, [104, Theorem 3.1].

Theorem 3.5. *Let X be a Banach space and $\mathcal{M}$ be any R-bounded B.a. of projections in X. Then the B.a. $\mathcal{M} \vee \mathcal{N}$ is bounded for every bounded B.a. of projections $\mathcal{N} \subseteq \mathcal{L}(X)$ which commutes with $\mathcal{M}$.*

The applicability of Theorem 3.5 stems from the fact that every Banach space X with property (α), a class of spaces introduced by G. Pisier, [109], has the property that *every* bounded B.a. of projections in X is automatically R-bounded, [104, Theorem 3.3]. We recall that *property* (α) holds if there exists a constant $\alpha \geq 0$ such that

$$\int_0^1 \int_0^1 \Big\| \sum_{j=1}^{m} \sum_{k=1}^{n} \varepsilon_{jk} r_j(s) r_k(t) x_{jk} \Big\|^2 \, ds \, dt$$
$$\leq \alpha^2 \int_0^1 \int_0^1 \Big\| \sum_{j=1}^{m} \sum_{k=1}^{n} r_j(s) r_k(t) x_{jk} \Big\|^2 \, ds \, dt$$

for every choice of $x_{jk} \in X$, $\varepsilon_{jk} \in \{-1, 1\}$ and for all $m, n \in \mathbb{N}$. It is shown in [109, Proposition 2.1] that every Banach space with l.u.st. and having finite cotype necessarily has property (α). In particular, every Banach lattice (which automatically has l.u.st., [34, Theorem 17.1]) with finite cotype has property (α). Actually, within the class of Banach spaces with l.u.st., having property (α) is equivalent to having finite cotype, [104, pp.488-489]. Concerning some relevant examples, note that c_0 and ℓ^∞ (for instance) have l.u.st. but fail to have finite cotype (and hence, fail to have property (α)). The von Neumann-Schatten ideals $\mathfrak{S}_p$, for $1 < p < \infty$, are Banach spaces with finite cotype but, for $p \neq 2$, fail to have property (α) (hence, also fail l.u.st.); see [65, Remark 2.10] and [104, Corollary 3.4]. For every $p > 2$, it is known that there exist closed subspaces of L^p (hence, they have property (α)) which fail to have l.u.st., [109, p.19].

For the definition and properties of Banach spaces which are *Gordon-Lewis spaces* (briefly, GL-spaces) we refer to [34, §17]. GL-spaces have property (α) iff they have finite cotype, [104, Theorem 4.4]. Every Banach space with l.u.st. is a GL-space but not conversely; see [104, p.491] for a discussion. Accordingly, the following result, [104, Theorem 4.2], is not subsumed by Theorem 3.4 above.

Theorem 3.6. *The product B.a. generated by any pair of commuting, bounded B.a.'s of projections in a GL-space X is again bounded.*

The particular B.a. $\mathrm{Bd}(X)$ consisting of all *band projections* in a Banach lattice X has the following remarkable property, [104, Theorem 5.8].

Theorem 3.7. *For a Banach lattice X the following are equivalent.*
(a) *X is Dedekind σ-complete and* $\mathrm{Bd}(X)$ *is R-bounded.*
(b) *X has finite cotype.*
(c) *X has property* (α).

A consequence of Theorem 3.7 and the discussion immediately after Theorem 3.5 is that *every* bounded B.a. of projections in a Dedekind σ-complete Banach lattice X is R-bounded precisely when *just* the B.a. $\mathrm{Bd}(X)$ is R-bounded! The techniques used in [104] to establish these facts are of interest in their own right and have further consequences. Given any Banach space X and any bounded B.a. of projections $\mathcal{M}$ in X, it is always possible to equip the cyclic space $\mathcal{M}[x]$ with a Banach lattice structure, for each $x \in X$; see [104, §6]. This leads to the following useful fact, [104, Proposition 6.3].

Theorem 3.8. *Let $\mathcal{M}$ be a bounded B.a. of projections in a Banach space X. Then $\overline{\mathcal{M}}_s$ is a Bade complete B.a. of projections iff each Banach lattice $\mathcal{M}[x]$, for $x \in X$, has order continuous norm.*

By applying the above criteria in each Banach lattice $\mathcal{M}[x]$, for $x \in X$, it turns out, for any Banach space X, that $\overline{\mathcal{M}}_s$ is Bade complete whenever $\mathcal{M}$ is R-bounded, [104, Theorem 6.6]. A consideration of the case $X = c_0$ and $\mathcal{M} = \mathrm{Bd}(c_0)$, which even has a cyclic vector, shows that the converse is false in general.

In conclusion we briefly mention some recent results in [106] on $C(K)$-representations which are also related to spectral measures and R-boundedness; see also [105]. For the following fact see [106, Proposition 2.17 & Remark 2.18].

Theorem 3.9. *Let K be a compact Hausdorff space and X be a Banach space. Let $\Phi : C(K) \to \mathcal{L}(X)$ be a continuous representation (i.e. linear and multiplicative) which is R-bounded, that is, the image under Φ of the unit ball of $C(K)$ is R-bounded in $\mathcal{L}(X)$. Then there exists a regular spectral measure $P : \mathcal{B}(K) \to \mathcal{L}_s(X)$ which is R-bounded (i.e. $P(\mathcal{B}(K)) \subseteq \mathcal{L}(X)$ is R-bounded) and satisfies*

$$\Phi(f) = \int_K f \, dP, \qquad f \in C(K). \tag{3.3}$$

Conversely, if $P : \mathcal{B}(K) \to \mathcal{L}_s(X)$ is any regular, R-bounded spectral measure, then Φ as defined by (3.3) *is a continuous, R-bounded representation.*

Let G be a lca group and consider $L^1(G)$ as a Banach algebra under convolution. A representation $\Psi : L^1(G) \to \mathcal{L}(X)$, always assumed to be continuous, is called *essential* if $\bigcup_{f\in L^1(G)} \Psi(f)(X)$ is dense in X. The next result (see [107]) follows by applying Theorem 3.9 to the dense subalgebra $\{\widehat{f} : f \in L^1(G)\}$ of $C_0(\Gamma)$ (the Banach space of all continuous functions on Γ which vanish at ∞) and by passing to $C(\Gamma_\infty)$, where Γ_∞ is the 1-point compactification of Γ.

Theorem 3.10. *Let $\Psi : L^1(G) \to \mathcal{L}(X)$ be an essential representation with $\{\Psi(f) : f \in L^1(G), \|\widehat{f}\|_\infty \le 1\}$ being R-bounded. Then there exists an R-bounded, regular spectral measure $P : \mathcal{B}(\Gamma) \to \mathcal{L}_s(X)$ with*

$$\Psi(f) = \int_\Gamma \widehat{f}\, dP, \qquad f \in L^1(G). \tag{3.4}$$

Conversely, given any R-bounded, regular spectral measure $P : \mathcal{B}(\Gamma) \to \mathcal{L}_s(X)$, the map Ψ defined by (3.4) *is an essential representation of $L^1(G)$ and $\{\int_\Gamma \widehat{f}\, dP : f \in L^1(G), \|\widehat{f}\|_\infty \le 1\}$ is R-bounded.*

For X a Banach lattice which satisfies either a lower p-estimate for some $1 \le p < 2$ or an upper q-estimate for some $2 < q \le \infty$, techniques are exhibited which can be used to decide about the R-boundedness of particular representations $\Psi : L^1(G) \to \mathcal{L}(X)$ defined on particular groups G; see [105; 107] for the details.

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