MULTIPLIERS FOR SPACES OF UNCONDITIONAL SERIES IN THE HARDY SPACES $H^p(\mathbb{D})$

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Let $H^p(\mathbb{D})$, $1 \leq p \leq \infty$, denote the usual Hardy spaces of analytic functions in the open unit disc $\mathbb{D}$. Consider the spaces H^p_{uc} consisting of all functions $f_a(z) = \sum_{n=0}^\infty a_n z^n$ such that $f_a \in H^p(\mathbb{D})$ and the convergence of the Taylor series of f_a is *unconditional* in $H^p(\mathbb{D})$. Here 'a' denotes the sequence of coefficients of $\sum_{n=0}^\infty a_n z^n$. Due to the Bounded Multiplier Test we have:

$$H^p_{uc} := \left\{ f_a(z) = \sum_{n=0}^\infty a_n z^n : f_b(z) = \sum_{n=0}^\infty b_n z^n \in H^p(\mathbb{D}), \forall\ |b_n| \leq |a_n| \right\}.$$

Some facts taken from [1], are as follows.

- The norm in H^p_{uc} is given by
$$\|f_a\|_{H^p_{uc}} := \sup_{|b_n|\leq|a_n|} \left\| \sum_{n=0}^\infty b_n z^n \right\|_{H^p(\mathbb{D})}.$$
- For $1 \leq p \leq 2$, due to a theorem or Orlicz, we have $H^p_{uc} = H^2(\mathbb{D})$.
- $H^\infty_{uc} = \ell^1(\mathbb{D})$, where $\ell^1(\mathbb{D}) := \{f_a(z) = \sum_{n=0}^\infty a_n z^n, a = (a_n) \in \ell^1\}$.
- For all even positive integers $p = 2k$ with $k \in \mathbb{N}$, due to the *majorant property* of Hardy and Littlewwod, we have the simpler description
$$H^{2k}_{uc} := \left\{ f_a(z) = \sum_{n=0}^\infty a_n z^n : f_{|a|}(z) = \sum_{n=0}^\infty |a_n| z^n \in H^{2k}(\mathbb{D}) \right\}$$
with the norm given by
$$\|f_a\|_{H^{2k}_{uc}} = \|f_{|a|}\|_{H^{2k}(\mathbb{D})} = \left\| \sum_{n=0}^\infty |a_n| z^n \right\|_{H^{2k}(\mathbb{D})}.$$
- For the particular case of H^4_{uc} the previous description becomes
$$H^4_{uc} = \left\{ f_a : f_{|a|} \in H^4(\mathbb{D}) \right\} = \left\{ f_a : f^2_{|a|} \in H^2(\mathbb{D}) \right\}$$
with

$$\|f_a\|_{H^4_{uc}} = \left\| f^2_{|a|} \right\|^{1/2}_{H^2(\mathbb{D})} = \left\| f_{|a|*|a|} \right\|^{1/2}_{H^2(\mathbb{D})} = \left(\sum_{n=0}^\infty \Big(\sum_{k=0}^n |a_k a_{n-k}| \Big)^2 \right)^{1/4}. \tag{1}$$

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Here we use the convolution of sequences. Namely, for $a = (a_n)_{n=0}^\infty$ and $b = (b_n)_{n=0}^\infty$, recall that

$$a * b = \Big((a*b)_n\Big)_{n=0}^\infty = \left(\sum_{k=0}^n a_k b_{n-k}\right)_{n=0}^\infty .$$

With this background we can now formulate the open problem. It concerns the determination of the space of all *analytic multipliers* of the space H^p_{uc}, that is, to identify all functions $\varphi_b(z) = \sum_{n=0}^\infty b_n z^n$, analytic in $\mathbb{D}$, such that the multiplication operator

$$M_{\varphi_b}\colon f_a \in H^p_{uc} \longmapsto \varphi_b \cdot f_a \in H^p_{uc}, \qquad \forall f_a \in H^p_{uc}, \tag{2}$$

is continuous. Label the set of all such multipliers by $\mathcal{M}(H^p_{uc})$. For $2 < p < \infty$ it is known that:

- $\ell^1(\mathbb{D}) \subset \mathcal{M}(H^p_{uc}) \subset H^\infty(\mathbb{D})$; see [1, Proposition 3.9].
- $A(\mathbb{D}) \not\subset \mathcal{M}(H^p_{uc})$, where $A(\mathbb{D})$ is the disc algebra; see [1, Proposition 3.10]. In particular, $\mathcal{M}(H^p_{uc}) \neq H^\infty(\mathbb{D})$, which contrasts to the known fact that $\mathcal{M}(H^p) = H^\infty(\mathbb{D})$ for $1 \le p \le \infty$.

In view of (1) and (2), the condition $\Phi(b) < \infty$, with

$$\Phi(b) := \sup_{a=(a_n)_{n=0}^\infty \neq 0} \frac{\sum\limits_{n=0}^\infty \left(\sum\limits_{k=0}^n \Big|\sum\limits_{j=0}^k a_j b_{k-j}\Big| \Big|\sum\limits_{j=0}^{n-k} a_j b_{n-k-j}\Big|\right)^2}{\sum\limits_{n=0}^\infty \Big(\sum\limits_{k=0}^n |a_k a_{n-k}|\Big)^2},$$

corresponds precisely to $\varphi_b \in \mathcal{M}(H^4_{uc})$.

<u>Question</u>: Is it the case that $\mathcal{M}(H^4_{uc}) = \ell^1(\mathbb{D})$?

If so, this means that $\Phi(b) < \infty$ should imply that $b \in \ell^1$. An answer would, of course, also throw some light on the more general problem of determining whether or not $\mathcal{M}(H^{2k}_{uc})$ is k-dependent (or $\mathcal{M}(H^p_{uc})$ is p-dependent).

References

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