À LA CARTE RECURRENCE RELATIONS FOR CONTINUOUS AND DISCRETE HYPERGEOMETRIC FUNCTIONS

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Abstract. We show how, using the constructive approach for special functions introduced by Nikiforov and Uvarov, one can obtain recurrence relations for the hypergeometric-type functions not only for the continuous case but also for the discrete and q-linear cases, respectively. Some applications in Quantum Physics are discussed.

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1. Introduction

Almost all Special Functions of Mathematical-Physics (e.g., Bessel, Airy, Weber, Whittaker, Gauss, Kummer and Hermite functions, the classical orthogonal polynomials, etc.) are solutions of the so-called *hypergeometric-type equation*

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0\,, \tag{1.1}$$

where σ and τ are polynomials with degrees not higher than two and one, respectively, and λ is a constant. In the present paper we will consider the *hypergeometric-type functions* $y = y_\nu(z)$ that are the solutions the equation (1.1) under the condition

$$\lambda + \nu\tau' + \frac{\nu(\nu-1)}{2}\sigma'' = 0,$$

where ν is a complex number. A basic important property of this class of functions is that *their derivatives are again hypergeometric-type functions.* The converse is also true when $\deg[\sigma(s)] = 2 \;\vee\; \deg[\tau(s)] = 1$: *any hypergeometric-type function is the derivative of a hypergeometric-type function.* More precisely [20]:

(1) if $y = y(z)$ is a solution of (1.1) then, the n-th derivative of $y(z)$, $v_n(z) := y^{(n)}(z)$, is a solution of

$$\sigma(z)v_n''(z) + \tau_n(z)v_n'(z) + \mu_n v_n(z) = 0, \tag{1.2}$$

where $\tau_n(z) = \tau(z) + n\sigma'(z)$, and $\mu_n = \mu_n(\lambda) = \lambda + n\tau' + \frac{n(n-1)}{2}\sigma''$;

(2) if $v_n(z)$ is a solution of (1.2) and $\mu_k \neq 0$ for $k = 1, \dots, n-1$, then $v_n = y^{(n)}(z)$ where $y = y(z)$ is a solution of (1.1).

Joining these two properties it is possible to derive many other ones as it is shown in [20] (see e.g. [12, 13, 14, 25] and references therein).

An interesting application of the aforesaid functions is the fact that, in most cases, the Schrödinger equation –that rules the quantum mechanical systems– for a wide class of potentials can be transformed into the equation (1.1) by an appropriate change of variables (see [20, Section 1]) and therefore, a deep knowledge of the Special Function Theory allow us to obtain several new relations for the wave functions (i.e. the solution of the stationary Schrödinger equation) as it is shown in the papers [6, 11].

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This method is also extensive to the *discrete case* when the hypergeometric equation (1.1) is changed by the second-order difference equation of hypergeometric type

$$\sigma(s)\nabla\Delta y(s)+\tau(s)\Delta y(s)+\lambda y(s)=0, \tag{1.3}$$

being $\sigma(s)$ and $\tau(s)$ polynomials of degree not greater than 2 and 1, respectively, λ is a constant, and $\Delta f(s)=f(s+1)-f(s)$ and $\nabla f(s)=\Delta f(s-1)$ are the forward and backward difference operators, respectively, or the q-case when (1.1) reads

$$\begin{gathered}\sigma(s)\frac{\Delta}{\Delta x(s-\frac{1}{2})}\left[\frac{\nabla y(s)}{\nabla x(s)}\right]+\tau(s)\frac{\Delta y(s)}{\Delta x(s)}+\lambda y(s)=0,\\ \sigma(s)=\widetilde{\sigma}(x(s))-\tfrac{1}{2}\widetilde{\tau}(x(s))\Delta x\left(s-\tfrac{1}{2}\right),\quad \tau(s)=\widetilde{\tau}(x(s)),\end{gathered} \tag{1.4}$$

where $\widetilde{\sigma}(x(s))$ and $\widetilde{\tau}(x(s))$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and λ is a constant. The last equation is usually called the *second order linear difference equation of hypergeometric type on non-uniform lattices.*

The last two equations modelize certain discrete systems such as the discrete oscillators [4, 10, 8], discrete Calogero-Sutherland model [19], q-analogues of the harmonic oscillator [2, 3, 8, 9], among others.

Here we will deal with the linear and q-linear lattices, respectively, i.e., lattices of the form

$$x(s)=c_1s+c_2 \quad \text{or} \quad x(s)=c_1(q)q^s+c_2(q). \tag{1.5}$$

Since the equation (1.4) is linear, one can restrict ourselves, without losing the generality, to the canonical lattices $x(s)=s$ and $x(s)=q^s$.

Notice that (1.4) becomes into (1.3) when $x(s)$ in the linear lattice $x(s)=s$ and the solutions of the difference equation are called the *discrete hypergeometric-type functions.* In the case of the q-linear lattice $x(s)=q^s$ the solutions of (1.4) are usually called the *q-special functions* on the q-linear lattice.

The recurrence relations for the special functions (and thus for the associated wave functions) are interesting not only from the theoretical point of view (they are useful for computing the values of matrix elements of certain physical quantities see e.g. [11] and references therein) but also they can be used to numerically compute the values of the functions as well as their derivatives, as it is shown in [6] for the case of Laguerre polynomials and the associated wave functions of the harmonic oscillator and the hydrogen atom. Nevertheless, we need to point out that, although the recurrence relations seem to be more useful for the evaluation of the corresponding functions than other direct methods, one should be very careful when using them (see e.g. the nice surveys [16, 23] on numerical evaluation and the convergence problem that appears when dealing with recurrence relations, or the most recent ones [17, 18] for the case of the hypergeometric function ${}_2F_1$).

2. The continuous case

In the following we will follow the notation of [20].

Before starting with the main results of this paper let us discuss a very important special case of the hypergeometric-type functions: the so-called hypergeometric polynomials. To obtain this family of particular solutions of (1.1) for a given λ we notice that when $\mu_n=0$, equation (1.2) has the particular solution $v_n(z)=C$, (constant). By (2), $v_n(z)=y^{(n)}$ where $y=y(z)$ is a solution of (1.1), i.e., when

$$\lambda=\lambda_n:=-n\tau'-\frac{n(n-1)}{2}\sigma'' \tag{2.1}$$

the equation (1.1) has a (particular) polynomial solution $y(z)=p_n(z)$, with $\deg[p_n(z)]=n$. Such polynomials are known as *polynomials of hypergeometric type* and correspond to the case when $\lambda = \lambda_n$ is given by (2.1). In particular, for them we have the *Rodrigues formula*

$$p_n(z) = \frac{B_n}{\rho(z)} \left[\sigma^n(z)\rho(z)\right]^{(n)},$$

where the B_n, $n = 0, 1, 2, \ldots$, are normalizing constants and $\rho(z)$ is a solution of the Pearson equation

$$\left[\sigma(z)\rho(z)\right]' = \tau(z)\rho(z)\,. \tag{2.2}$$

Assuming that ρ is an analytic function on and inside a closed contour $\mathcal{C}$ surrounding the point $s = z$ and making use of the Cauchy's integral theorem we may write the Rodrigues formula in the following integral form

$$y_n(z) = \frac{C_n}{\rho(z)} \int_{\mathcal{C}} \frac{\sigma^n(s)\rho(s)}{(s-z)^{(n+1)}} ds\,,$$

where the $C_n = n! B_n/(2\pi i)$ is a normalizing constant and $\rho(z)$ satisfies (2.2).

The above integral representation is extremely useful for obtaining algebraic and asymptotic formulas for the hypergeometric polynomials (see e.g. [20] and [15], and references therein). Moreover, any solution of the hypergeometric equation (1.1) admit a similar integral representation. In fact, the following theorem holds:

Theorem 2.1. [20, page 10] *Let $\rho(z)$ satisfy the Pearson equation (2.2) where ν is a solution of (2.4) and let D be a region of the complex plane that contains the piecewise smooth curve $\mathcal{C}$ of finite length. Then, equation (1.1) has a particular solution of the form*[1]

$$y_\nu(z) = \frac{C_\nu}{\rho(z)} \int_{\mathcal{C}} \frac{\sigma^\nu(s)\rho(s)}{(s-z)^{(\nu+1)}} ds\,, \tag{2.3}$$

where C_ν is a normalizing constant and ν is an arbitrary complex parameter connected with λ by

$$\lambda = \lambda_\nu = -\nu\tau' - \frac{\nu(\nu-1)}{2}\sigma''\,. \tag{2.4}$$

provided that the functions $\frac{\sigma^\nu(s)\rho(s)}{(s-z)^{(\nu+k)}}$, for $k = 1, 2$, are continuous as functions of the variables $s \in \mathcal{C}$, $z \in D$; and that for each fixed $s \in \mathcal{C}$, they are analytic as functions of $z \in D$. Here $\mathcal{C}$ is such that $\left.\frac{\sigma^{\nu+1}(s)\rho(s)}{(s-z)^{\nu+2}}\right|_{s_1}^{s_2} = 0$, where s_1 and s_2 are the endpoints of $\mathcal{C}$.

In the following we will use the notation

$$\tau_\nu(z) = \tau(z) + \nu\sigma'(z) = \tau'_\nu\, z + \tau_\nu(0), \quad \nu \in \mathbb{C}$$

and we will restrict ourselves to the condition $\deg[\sigma(s)] = 2$ or $\deg[\tau(s)] = 1$. For the hypergeometric functions y_ν the following theorem holds [20, page 18]:

Theorem 2.2. *Let $y_{\nu_i}^{(k_i)}(z)$, $i = 1, 2, 3$, be any three derivatives of order k_i of the functions of hypergeometric type being $\nu_i - \nu_j$ an integer and such that*

$$\left.\frac{\sigma^{\nu_0+1}(s)\rho(s)}{(s-z)^{\mu_0}} s^m\right|_{s_1}^{s_2} = 0 \quad , m = 0, 1, 2, \ldots\,,$$

where ν_0 denotes the index ν_i with minimal real part and μ_0 the one with maximal real part. Then, there exist three non vanishing polynomials $B_i(z)$, $i = 1, 2, 3$, such that

$$\sum_{i=1}^{3} B_i(z) y_{\nu_i}^{(k_i)}(z) = 0. \tag{2.5}$$

[1]If the integral in (2.3) is an improper one, then the result remains valid if the convergence of the integral is uniform.

In the similar fashion we can obtain the following Theorem (for the proof see [12]) for certain four-term recurrence relations:

Theorem 2.3. *Consider the functions of hypergeometric type* $y_{\nu-1}^{(k)}(z)$, $y_\nu^{(k)}(z)$, $y_\nu^{(k+1)}(z)$ *and* $y_{\nu+1}^{(k)}(z)$. *Suppose that* $\rho(z)$ *is a solution of (2.2), satisfying the condition*

$$\left.\frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu-k-1}}s^m\right|_{s_1}^{s_2}=0,\quad m=0,1,2,\dots,$$

where s_1 *and* s_2 *are the end points of* $\mathcal{C}$. *Then, there exist polynomial coefficients* $D_{ik}(z)$, $i=1,2,3,4$, *not all identically zero, such that*

$$D_{1k}(z)y_{\nu-1}^{(k)}(z)+D_{2k}(z)y_\nu^{(k)}(z)+D_{3k}(z)y_\nu^{(k+1)}(z)+D_{4k}(z)y_{\nu+1}^{(k)}(z)=0. \tag{2.6}$$

Moreover, the functions D_{ik}, $i=1,2,3,4$ *are given by*

$$\begin{cases} D_{1k}=\dfrac{C_\nu}{C_{\nu-1}}\tau'_\nu\tau'_{\frac{\nu+k-1}{2}}\tau'_{\frac{\nu+k-2}{2}}\left(H(z)-\tau'_{\nu-\frac12}\right)\times \\ \qquad\qquad \left[\tau^2_{\nu-1}(0)\frac{\sigma''}{2}+\tau'_{\nu-1}\Big(\sigma(0)\tau'_{\nu-1}-\sigma'(0)\tau_{\nu-1}(0)\Big)\right], \\ D_{2k}=(\nu-k)\tau'_{\nu-\frac12}\tau'_\nu\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu+k-1}{2}}\times \\ \qquad \left[\Big(\tau(0)\sigma''-\sigma'(0)\tau'\Big)-\dfrac{\sigma''}{2}\tau_{\nu-1}(z)+H(z)\dfrac{\sigma'(0)\tau'_\nu-\tau_\nu(0)\sigma''}{\tau'_\nu}\right], \\ D_{3k}=-\tau'_{\nu-1}\tau'_{\nu-\frac12}\tau'_{\frac{\nu}{2}-1}\tau'_\nu\left(H(z)-\tau'_{\frac{\nu+k-1}{2}}\right)\sigma(z), \\ D_{4k}=H(z)(\nu-k)(\nu-k+1)\dfrac{\sigma''}{2}\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\nu-1}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}, \end{cases}$$

where $H(z)$ *is an arbitrary function of* z.

Choosing in the last formula the function $H(z)$ appropriately, one can obtain several recurrence relations of 4th and 3rd order, respectively as it is shown in [12]. For example, if we put $H(z)=\tau'_{\frac{\nu+k-1}{2}}$, then $D_{3k}=0$ and, after some simplifications one obtain the following result that was firstly obtained in [13] (see also [12])

Theorem 2.4.

$$J_{1k}(z)y_{\nu-1}^{(k)}(z)+J_{2k}(z)y_\nu^{(k)}(z)+J_{3k}(z)y_{\nu+1}^{(k)}(z)=0,$$

where

$$\begin{cases} J_{1k}(z)=\dfrac{C_\nu}{C_{\nu-1}}\tau'_{\frac{\nu+k-2}{2}}\tau'_\nu\left[\tau'_{\nu-1}\left(\sigma'(0)\tau_{\nu-1}(0)-\sigma(0)\tau'_{\nu-1}\right)-\tau^2_{\nu-1}(0)\dfrac{\sigma''}{2}\right], \\ J_{2k}(z)=-\tau'_{\frac{\nu}{2}-1}\tau'_{\nu-\frac12}\left[\tau'_\nu\tau'_{\nu-1}z+\tau'\tau_{2\nu-k}(0)+\sigma''\left(k\tau(0)-\tau_{\nu(1-\nu)}(0)\right)\right], \\ J_{3k}(z)=(\nu-k+1)\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\tau'_{\nu-1}. \end{cases}$$

Notice that it gives the coefficients for the standard three-term recurrence relation for the derivative of k-order of the hypergeometric function. The case $k=0$ was considered in the paper [24]. Obviously the last theorem is a particular case of the Theorem 2.2.

Examples. As an example we will derive the relations obtained from the Theorems 2.3 and 2.4 applied to the *hypergeometric* $F(\alpha,\beta,\gamma,z)$, *confluent hypergeometric* $F(\alpha,\gamma,z)$, and *Hermite* $H_\nu(z)$ functions. The explicit expression for these functions can be found, e.g., in [20, §20, section 2. page 258]. In terms of the generalized hypergeometric notation the first two correspond to ${}_2F_1(\alpha,\beta;\gamma;z)$ and ${}_1F_1(\alpha;\gamma;z)$, respectively. We will take, for these functions, the normalization considered in [20, §20, section 2. page 255]. Since these three functions satisfy the following differential linear equations of type (1.1)

- deg $\big(\sigma(z)\big) = 2$, the hypergeometric functions $u(z) = F(\alpha, \beta, \gamma, z)$
$$z(1-z)\,u'' + \big[\gamma - \big(\alpha+\beta+1\big)z\big]\,u' - \alpha\beta\,u = 0\,; \tag{2.7}$$
i.e., the equation with
$$\sigma(z) = z(1-z), \qquad \tau(z) = \gamma - \big(\alpha+\beta+1\big)z, \qquad \lambda = -\alpha\beta\,. \tag{2.8}$$
- deg $\big(\sigma(z)\big) = 1$, the confluent hypergeometric functions $u(z) = F(\alpha, \gamma, z)$
$$z\,u'' + \big(\gamma - z\big)\,u' - \alpha\,u = 0\,; \tag{2.9}$$
i.e., the equation (1.1) with
$$\sigma(z) = z, \qquad \tau(z) = \gamma - z, \qquad \lambda = -\alpha\,. \tag{2.10}$$
- deg $\big(\sigma(z)\big) = 0$, the Hermite functions $u(z) = H_\nu(z)$
$$u'' - 2z\,u' + 2\nu\,u = 0\,, \tag{2.11}$$
i.e., the equation (1.1) with
$$\sigma(z) = 1, \qquad \tau(z) = -2z, \qquad \lambda = 2\nu\,. \tag{2.12}$$

we can use Theorem 2.3 we obtain the relations [12]. For the sake of completeness we summarize summarize the results here:

▶ For the *Hypergeometric Equation*, choosing $\nu = -\alpha$ or $\nu = -\beta$, the recurrence relation (2.6) holds with

$$\begin{cases} D_{1k}(z) = -\alpha(\alpha-1)(\beta+k)(\beta+k-1)(\beta-\gamma)(\beta-\alpha+1)\big[R(z) - (\beta-\alpha)\big] \\ D_{2k}(z) = (\alpha-1)(\alpha+k)(\beta-1)(\beta+k)(\beta-\alpha)\times \\ \qquad\qquad \Big\{\big[(\beta-\gamma) - (\beta-\alpha-1)z\big](\beta-\alpha+1) + R(z)(\beta-3\alpha+2\gamma+1\big)\Big\} \\ D_{3k}(z) = (\alpha-1)(\beta-1)(\beta-\alpha-1)(\beta-\alpha)(\beta-\alpha+1)\Big(R(z) - (\beta+k)\Big)z(1-z) \\ D_{4k}(z) = R(z)(\alpha+k)(\alpha+k-1)\beta(\beta-1)(\gamma-\alpha)(\beta-\alpha-1) \end{cases}$$

being $R(z)$ is an arbitrary function (polynomial) of z.

▶ For the *Confluent Hypergeometric Equation*, choosing $\nu = -\alpha$, (2.6) holds with

$$D_{1k}(z) = -\alpha, \quad D_{2k}(z) = \alpha+k, \quad D_{3k}(z) = z, \quad D_{4k}(z) = 0.$$

▶ Finally, for the *Hermite Equation*, and for an arbitrary ν the relation (2.6) holds with

$$D_{1k}(z) = 2\nu, \quad D_{2k}(z) = 0, \quad D_{3k}(z) = -1, \quad D_{4k}(z) = 0.$$

In a similar fashion, but using Theorem 2.4 we find the following recurrence relations satisfied by the *hypergeometric*, *confluent hypergeometric*, and *Hermite* functions, respectively:

▶ Hypergeometric function

$$\begin{aligned} &\big(\beta-1\big)\big(\gamma-\alpha\big)\big(\beta-\alpha-1\big)\,{}_2F_1(\alpha-1,\beta;\gamma;z)+ \\ &(\beta-\alpha)\Big\{\big[(\beta-\alpha)^2-1\big]z - \big(\alpha+\beta+1\big)\big(\gamma-2\alpha\big) + 2\big(\gamma-\alpha(\alpha+1)\big)\Big\}\,{}_2F_1(\alpha,\beta;\gamma;z)+ \\ &\alpha\big(\beta-\alpha+1\big)\big(\gamma-\beta\big)\,{}_2F_1(\alpha+1,\beta;\gamma;z) = 0\,. \end{aligned}$$

▶ Confluent hypergeometric function

$$\big(\gamma-\alpha\big)\,{}_1F_1(\alpha-1;\gamma;z) + \Big[z - \big(\gamma-2\alpha\big)\Big]\,{}_1F_1(\alpha;\gamma;z) - \alpha\,{}_1F_1(\alpha+1;\gamma;z) = 0.$$

▶ Hermite function

$$H_{\nu+1}(z) - 2z\, H_\nu(z) + 2\nu\, H_{\nu-1}(z) = 0.$$

To conclude this section let us mention that the above theorems can be used for obtaining several recurrence relations for the orthogonal polynomials of hypergeometric type as it has been done in [12].

2.1. **Application to Quantum Mechanics.** Let us show two applications to Quantum Physics. For more details see [11] and references therein.

The N-dimensional isotropic harmonic oscillator (I.H.O.) is described by the Schrödinger equation

$$\left(-\Delta + \frac{1}{2}\lambda^2 r^2\right)\Psi = E\Psi, \qquad \Delta = \sum_{k=1}^{n} \frac{\partial}{\partial x_k}, \quad r = \sqrt{\sum_{k=1}^{n} x_k^2}\,.$$

For solving it one uses the method of separation of variables that leads to a solution of the form $\Psi = R_{nl}^{(N)}(r) Y_{lm}(\Omega_N)$, where $R_{nl}^{(N)}(r)$ is the radial part, usually called the *radial wave functions*, defined by

$$R_{nl}^{(N)}(r) = (\mathcal{N})_{nl}^{(N)}\; r^l e^{-\frac{1}{2}\lambda r^2} L_n^{l+\frac{N}{2}-1}(\lambda r^2), \tag{2.13}$$

where

$$(\mathcal{N})_{nl}^{(N)} = \sqrt{\frac{2n!\lambda^{l+\frac{N}{2}}}{\Gamma\left(n+l+\frac{N}{2}\right)}},$$

being $n = 0, 1, 2, \dots$ and $l = 0, 1, 2, \dots$, the quantum numbers, and $N \geq 3$ the dimension of the space. The angular part $Y_{lm}(\Omega_N)$ are the so-called Nth-spherical or hyperspherical harmonics [20]. In the following, we will assume that the parameters n, l, N are nonnegative integers.

We will look, now, for recurrence relations connecting three different radial functions.

Theorem 2.5. [11] *Let $R_{nl}^{(N)}(r)$, $R_{n+n_1,l+l_1}^{(N)}(r)$ and $R_{n+n_2,l+l_2}^{(N)}(r)$ be three different radial functions of the N-th dimensional isotropic harmonic oscillator, were $n_1\,, n_2$ and $l_1\,, l_2$ are integers such that*

$$\min\left(n+n_1, n+n_2, l+l_1, l+l_2\right) \geq 0.$$

Then, there exist non-vanishing polynomials in r, A_0, A_1, and A_2, such that

$$A_0 R_{n,l}^{(N)}(r) + A_1 R_{n+n_1,l+l_1}^{(N)}(r) + A_2 R_{n+n_2,l+l_2}^{(N)}(r) = 0. \tag{2.14}$$

Proof: For the sake of simplicity we will prove the theorem for the case when $l_1\,, l_2$ are nonnegative integers. The case when $l_1\,, l_2$ are *integers* can be derived in the same way since the "cases" $R_{nl}^{(N)}$, $R_{n+n_1,l\pm l_1}^{(N)}$, $R_{n+n_2,l\pm l_2}^{(N)}$ can be reduced to $R_{nl}^{(N)}$, $R_{n+n_1,l+l_1}^{(N)}$, $R_{n+n_2,l+l_2}^{(N)}$ by choosing $l = \min\left(l, l\pm l_1, l\pm l_2\right)$.

From (2.13) we have

$$L_n^{l+\frac{N}{2}-1}(\lambda r^2) = \left((\mathcal{N})_{n,l}^{(N)}\right)^{-1} r^{-l} e^{\frac{1}{2}\lambda r^2} R_{nl}^{(N)}(r). \tag{2.15}$$

Thus,

$$L_{n+n_1}^{(l+l_1)+\frac{N}{2}-1}(\lambda r^2) = \left((\mathcal{N})_{n+n_1,l+l_1}^{(N)}\right)^{-1} r^{-(l+l_1)} e^{\frac{1}{2}\lambda r^2} R_{n+n_1,l+l_1}^{(N)}(r), \tag{2.16}$$

$$L_{n+n_2}^{(l+l_2)+\frac{N}{2}-1}(\lambda r^2) = \left((\mathcal{N})_{n+n_2,l+l_2}^{(N)}\right)^{-1} r^{-(l+l_2)} e^{\frac{1}{2}\lambda r^2} R_{n+n_2,l+l_2}^{(N)}(r). \tag{2.17}$$

Putting $s = \lambda r^2$, it is possible to rewrite the left hand side of (2.16) and (2.17) in the form

$$L_{n+n_1}^{(l+l_1)+\frac{N}{2}-1}(s) = (-1)^{l_1}\frac{d^{l_1}}{ds^{l_1}}L_{n+n_1+l_1}^{l+\frac{N}{2}-1}(s),, \tag{2.18}$$

$$L_{n+n_2}^{(l+l_2)+\frac{N}{2}-1}(s) = (-1)^{l_2}\frac{d^{l_2}}{ds^{l_2}}L_{n+n_2+l_2}^{l+\frac{N}{2}-1}(s), \tag{2.19}$$

respectively. On the other hand, by the generalized three-term recurrence relation (2.5), there exist non-vanishing polynomials $B_i(s)$, $i = 0, 1, 2$, such that

$$B_0 L_n^{l+\frac{N}{2}-1}(s) + B_1\frac{d^{l_2}}{ds^{l_2}}L_{n+n_1+l_1}^{l+\frac{N}{2}-1}(s) + B_2\frac{d^{l_1}}{ds^{l_1}}L_{n+n_2+l_2}^{l+\frac{N}{2}-1}(s) = 0.$$

Thus, since (2.18) and (2.19),

$$C_0 L_n^{l+\frac{N}{2}-1}(s) + C_1 L_{n+n_1}^{(l+l_1)+\frac{N}{2}-1}(s) + C_2 L_{n+n_2}^{(l+l_2)+\frac{N}{2}-1}(s) = 0, \tag{2.20}$$

where $C_0 = B_0$, $C_1 = (-1)^{l_1}B_1$ and $C_2 = (-1)^{l_2}B_2$. If we substitute (2.15)–(2.17) in (2.20), we have

$$\left((\mathcal{N})_{n,l}^{(N)}\right)^{-1} C_0 R_{nl}^{(N)}(r) + \left((\mathcal{N})_{n+n_1,l+l_1}^{(N)}\right)^{-1} C_1 r^{-l_1} R_{n+n_1,l+l_1}^{(N)}(r) + \left((\mathcal{N})_{n+n_2,l+l_2}^{(N)}\right)^{-1} C_2 r^{-l_2} R_{n+n_2,l+l_2}^{(N)}(r) = 0,$$

which transforms into (2.14) where

$$A_0 = \left((\mathcal{N})_{n,l}^{(N)}\right)^{-1} C_0 r^{l_1+l_2}, \quad A_1 = \left((\mathcal{N})_{n+n_1,l+l_1}^{(N)}\right)^{-1} C_1 r^{l_2}$$

and

$$A_2 = \left((\mathcal{N})_{n+n_2,l+l_2}^{(N)}\right)^{-1} C_2 r^{l_1}.$$

Obviously these functions A_i, $i = 0, 1, 2$, are polynomials in r. ■

In a similar way we have the following result for the so-called ladder operators for the radial wave functions:

Theorem 2.6. [11] *Let $R_{n,l}^{(N)}(r)$ and $R_{n+n_1,l+l_1}^{(N)}(r)$ be two radial functions of the N-th dimensional isotropic harmonic oscillator and let* $\min(n + n_1, l + l_1) \geq 0$ *and $(n_1)^2 + (l_1)^2 \neq 0$, where n_1 and l_1 are integers. Then, there exist not vanishing polynomials in r, A_0, A_1, and A_2, such that*

$$A_0 R_{n,l}^{(N)}(r) + A_1\frac{d}{dr}R_{n,l}^{(N)}(r) + A_2 R_{n+n_1,l+l_1}^{(N)}(r) = 0.$$

An important instances of the ladder operators are the creation and annihilation operators of Quantum Mechanics which can be obtained from the above formula when $n_1 = \pm 1$ and $l_1 = 0$.

Using the above theorem and the properties of Laguerre polynomials in [6, 11] a lot of recurrence relations and ladder operators have been obtained. We will show here two of special interest:

► Putting in Theorem 2.5 $n_1 = -1$, $n_2 = 1$, $l_1 = l_2 = 0$ we obtain [11, page 2059]

$$\sqrt{n\left(n+l+\frac{N}{2}-1\right)}R_{n-1,l}^{(N)}(r) + \left[\lambda r^2 - \left(2n+l+\frac{N}{2}\right)\right]R_{n,l}^{(N)}(r) + \sqrt{(n+1)\left(n+l+\frac{N}{2}\right)}R_{n+1,l}^{(N)}(r) = 0. \tag{2.21}$$

This is an standard three term recurrence relation that is seem to be the most effective way of computing numerically the values of the radial wave functions of the isotropic harmonic oscillator [6].

► The other one follows from Theorem 2.6 when $n_1 = 0$, $l_1 = 1$ [11, page 2061]

$$\left[\frac{d}{dr} - \lambda r - \frac{l}{r}\right] R_{n,l}^{(N)}(r) = -2\sqrt{\lambda\left(n + l + \frac{N}{2}\right)} R_{n,l+1}^{(N)}(r). \tag{2.22}$$

This relation joint with (2.21) allow us to compute efficiently the derivatives of the radial wave functions of the isotropic harmonic oscillator [6].

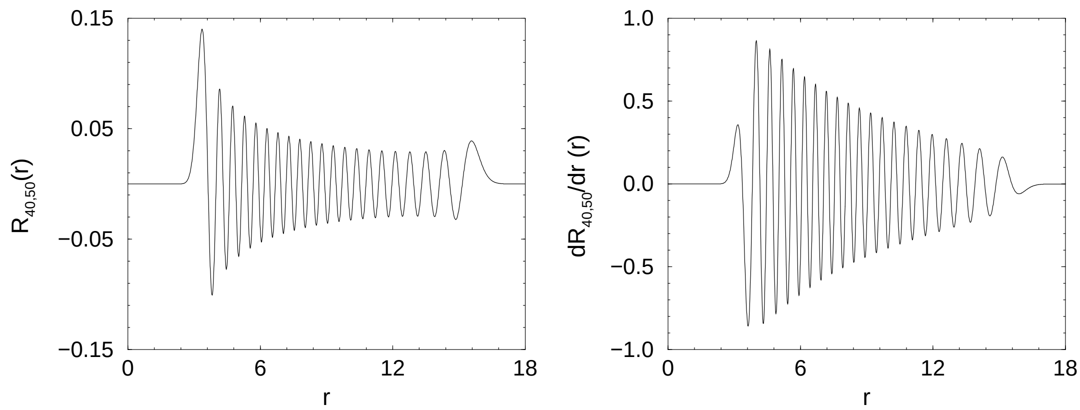

FIGURE 1. In the left panel we show the $R_{40,50}(r)$ computed by using the recurrence relation (2.21). In the right panel we represent the derivative of this function computed from the ladder-type relation (2.22).

A similar analysis have been done not only for the Radial functions of the Hydrogen atom [6, 11] but also for all the exactly solvable models which wave functions are written in terms of the hypergeometric type functions.

3. The discrete case

Let us consider now the case of the hypergeometric-type equation (1.4) on linear lattices (1.5). The equation (1.4) can be rewritten in the symmetric form

$$\frac{\Delta}{\Delta x(s-\frac{1}{2})}\left[\sigma(s)\rho(s)\frac{\nabla y(s)}{\nabla x(s)}\right] + \lambda_n \rho(s) y(s) = 0,$$

where $\rho(s)$ and $\rho_k(s)$ are the weight functions satisfying the Pearson-type difference equations $\triangle\left[\sigma(s)\rho(s)\right] = \tau(s)\rho(s)\Delta x(s-\frac{1}{2})$.

Next, we define the *k-order difference derivative* of a solution $y(s)$ of (1.4)

$$y^{(k)}(s) := \Delta^{(k)}[y(s)] = \frac{\Delta}{\Delta x_{k-1}(s)}\frac{\Delta}{\Delta x_{k-2}(s)}\cdots\frac{\Delta}{\Delta x(s)}[y(s)],$$

where $x_\nu(s) = x(s+\frac{\nu}{2})$. It is known [1, 21] that $y^{(k)}(s)$ also satisfy a difference equation of the same type. In fact, for the solutions of the difference equation (1.4) the following theorem, which is the discrete analog of 2.1, holds

Theorem 3.1. [22, Th. 2.2] *The difference equation (1.4) has a particular solution of the form*

$$y_\nu(z) = \frac{C_\nu}{\rho(z)}\sum_{s=a}^{b-1}\frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s) - x_\nu(z)]^{(\nu+1)}} \tag{3.1}$$

if the condition

$$\left.\frac{\sigma(x)\rho_\nu(x)\nabla x_{\nu+1}(s)}{[x_{\nu-1}(s)-x_{\nu-1}(z+1)]^{(\nu+1)}}\right|_a^b=0, \tag{3.2}$$

is satisfied, and of the form

$$y_\nu(z)=\frac{C_\nu}{\rho(z)}\int_C\frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s)-x_\nu(z)]^{(\nu+1)}}ds \tag{3.3}$$

if the condition

$$\int_C\Delta_s\frac{\sigma(x)\rho_\nu(x)\nabla x_{\nu+1}(s)}{[x_{\nu-1}(s)-x_{\nu-1}(z+1)]^{(\nu+1)}}=0 \tag{3.4}$$

is satisfied. Here C is a contour in the complex plane, C_ν is a constant, $\rho(s)$ and $\rho_\nu(s)$ are the solution of the Pearson-type equations

$$\begin{aligned}\frac{\rho(s+1)}{\rho(s)}&=\frac{\sigma(s)+\tau(s)\Delta x(s-\frac12)}{\sigma(s+1)}=\frac{\phi(s)}{\sigma(s+1)},\\ \frac{\rho_\nu(s+1)}{\rho_\nu(s)}&=\frac{\sigma(s)+\tau_\nu(s)\Delta x_\nu(s-\frac12)}{\sigma(s+1)}=\frac{\phi_\nu(s)}{\sigma(s+1)},\end{aligned} \tag{3.5}$$

where $x_\nu(s)=x(s+\nu/2)$,

$$\tau_\nu(s)=\frac{\sigma(s+\nu)-\sigma(s)+\tau(s+\nu)\Delta x(s+\nu-\frac12)}{\Delta x_{\nu-1}(s)},$$

ν is the root of the equation

$$\lambda_\nu+[\nu]_q\left\{\alpha_q(\nu-1)\widetilde{\tau}'+[\nu-1]_q\frac{\widetilde{\sigma}''}{2}\right\}=0,$$

and $[\nu]_q$ and $\alpha_q(\nu)$ are the q-numbers

$$[\nu]_q=\frac{q^{\nu/2}-q^{-\nu/2}}{q^{1/2}-q^{-1/2}},\qquad \alpha_q(\nu)=\frac{q^{\nu/2}+q^{-\nu/2}}{2},\quad \forall\nu\in\mathbb{C}, \tag{3.6}$$

respectively. The generalized powers $[x_k(s)-x_k(z)]^{(\nu)}$ are defined by

$$[x_k(s)-x_k(z)]^{(\nu)}=(q-1)^\nu c_1^\nu q^{\nu(k-\nu+1)/2}q^{\nu z}\frac{\Gamma_q(s-z+\nu)}{\Gamma_q(s-z)},\quad \nu\in\mathbb{R},$$

for the exponential lattice $x(s)=c_1q^s+c_2$ and

$$[x_k(s)-x_k(z)]^{(\nu)}=c_1^\nu\frac{\Gamma(s-z+\mu)}{\Gamma(s-z)},\quad \nu\in\mathbb{R},$$

for the linear lattice $x(s)=c_1s+c_2$, respectively.

Let us define the functions[2]

$$\Phi_{\nu,\mu}(z)=\sum_{s=a}^{b-1}\frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s)-x_\nu(z)]^{(\mu+1)}} \tag{3.7}$$

and

$$\Phi_{\nu,\mu}(z)=\int_C\frac{\rho_\nu(s)\nabla x_{\nu+1}(s)}{[x_\nu(s)-x_\nu(z)]^{(\nu+1)}}ds. \tag{3.8}$$

Notice that y_ν and $\Phi_{\nu,\mu}$ are related by the formula

$$y_\nu(z)=\frac{C_\nu}{\rho(z)}\Phi_{\nu,\nu}(z).$$

The following straightforward result, which proof we will omit, holds

[2]Obviously the functions (3.7) correspond to the boundary condition (3.2), whereas the functions (3.8) correspond to the condition (3.4).

Lemma 3.2. *For the functions $\Phi_{\nu,\mu}(z)$ the following relation holds*

$$\nabla_z \, \Phi_{\nu,\mu}(z) = [\mu+1]_q \nabla x_{\nu-\mu}(z) \Phi_{\nu,\mu+1}(z), \tag{3.9}$$

where $[t]_q$ denotes the symmetric q-numbers (3.6).

From (3.9) follows that $\Delta_z \, \Phi_{\nu,\mu}(z) = [\mu+1]_q \Delta x_{\nu-\mu}(z) \Phi_{\nu,\mu+1}(z+1)$.

In this case, in a similar fashion as for the continuous case, we can prove the following

Lemma 3.3. *Let $x(z)$ be $x(z) = q^z$ or $x(z) = z$. Then, any three functions $\Phi_{\nu_i,\mu_i}(z)$, $i = 1, 2, 3$, are connected by a linear relation*

$$\sum_{i=1}^{3} A_i(z) \Phi_{\nu_i,\mu_i}(z) = 0, \tag{3.10}$$

with non-zero at the same time polynomial coefficients on $x(z)$, $A_i(z)$, provided that the differences $\nu_i - \nu_j$ and $\mu_i - \mu_j$, $i, j = 1, 2, 3$, are integers and that the following condition holds[3]

$$\left. \frac{x^k(s)\sigma(s)\rho_{\nu_0}(s)}{[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)]^{(\mu_0)}} \right|_{s=a}^{s=b} = 0, \quad k = 0, 1, 2, \ldots, \tag{3.11}$$

and

$$\int_C \Delta_s \frac{x^k(s)\sigma(s)\rho_{\nu_0}(s) \, ds}{[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)]^{(\mu_0)}} = 0, \quad k = 0, 1, 2, \ldots, \tag{3.12}$$

when the functions Φ_{ν_i,μ_i} are given by (3.7). Here ν_0 is the ν_i, $i = 1, 2, 3$, with the smallest real part and μ_0 is the μ_i, $i = 1, 2, 3$, with the largest real part.

The above Lemma when $q \to 1$ and $x(s) = s$ leads to the corresponding result on the uniform lattice $x(s)$ (see [5, Lemma 3.1, page 885]).

In [22] the following relevant relation was established

$$\Delta^{(k)} y_\nu(s) = \frac{C_\nu^{(k)}}{\rho_k(s)} \Phi_{\nu\,\nu-k}(s), \tag{3.13}$$

where

$$C_\nu^{(k)} = C_\nu \prod_{m=0}^{k-1} \left[\alpha_q(\nu+m-1)\widetilde{\tau}' + [\nu+m-1]_q \frac{\widetilde{\sigma}''}{2} \right].$$

This relation is valid for solutions of the form (3.1) and (3.3) of the difference equation (1.4).

In the following, $y_n^{(k)}(s)$ denotes the k-th differences $\Delta^{(k)} y_n(s)$.

Theorem 3.4. *In the same conditions as in Lemma 3.3, any three functions $y_{\nu_i}^{(k_i)}(s)$, $i = 1, 2, 3$, are connected by a linear relation*

$$\sum_{i=1}^{3} B_i(s) y_{\nu_i}^{(k_i)}(s) = 0, \tag{3.14}$$

where the $B_i(s)$, $i = 1, 2, 3$, are polynomials.

[3]Condition (3.11) corresponds to the functions (3.7) whereas (3.12) corresponds to (3.8). In some cases theses conditions are equivalent to the condition $x(s)^k \sigma(s) \rho_{\nu_0}(s)|_{s=a}^{s=b} = 0$, $k = 0, 1, 2, \ldots$.

Proof. From Lemma 3.3 we know that there exists three polynomials $A_i(s)$, $i=1,2,3$ such that

$$\sum_{i=1}^{3} A_i(s)\Phi_{\nu_i,\nu_i-k_i}(s)=0,$$

then, using the relation (3.13), we find

$$\sum_{i=1}^{3} A_i(s)(C_\nu^{(k)})^{-1}\rho_{k_i}(s)y_{\nu_i}^{(k_i)}(s)=0.$$

Now, dividing the last expression by $\rho_{k_0}(s)$, where $k_0=\min\{k_1,k_2,k_3\}$, and using the identity $\rho_{\nu_i}(s)=\phi(s+\nu_0)\phi(s+\nu_0+1)\dots\phi(s+\nu_i-1)\rho_{\nu_0}(s)$, which is a consequence of the Pearson equation (3.5), we obtain

$$\sum_{i=1}^{3} B_i(s)y_{\nu_i}^{(k_i)}(s)=0,\quad B_i(s)=A_i(s)(C_\nu^{(k)})^{-1}\phi(s+k_0)\cdots\phi(s+k_i-1),$$

which completes the proof. □

Corollary 3.5. *In the same conditions as in Lemma 3.3, the following* Δ*-ladder-type relation hold*

$$B_1(s)y_\nu(s)+B_2(s)\frac{\Delta y_\nu(s)}{\Delta x(s)}+B_3(s)y_{\nu+m}(s)=0,\qquad m\in\mathbb{Z},\tag{3.15}$$

with polynomials coefficients $B_i(s)$, $i=1,2,3$.

Proof. It is sufficient to put $k_1=k_3=0$, $k_2=1$, $\nu_1=\nu_2=\nu$ and $\nu_3=\nu+m$ in (3.14). □

Notice that for the case $m=\pm1$ (3.15) becomes

$$B_1(s)y_\nu(s)+B_2(s)\frac{\Delta y_\nu(s)}{\Delta x(s)}+B_3(s)y_{\nu+1}(s)=0,\tag{3.16}$$

$$\widetilde{B}_1(s)y_\nu(s)+\widetilde{B}_2(s)\frac{\Delta y_\nu(s)}{\Delta x(s)}+\widetilde{B}_3(s)y_{\nu-1}(s)=0,\tag{3.17}$$

with polynomials coefficients $B_i(s)$ and $\widetilde{B}_i(s)$, $i=1,2,3$. The above relations are usually called raising and lowering operators, respectively, for the functions y_ν.

From the above expressions, it is easy to obtain the raising and lowering operators for the functions y_ν associated to the $\nabla/\nabla x(s)$ operators:

$$C_1(s)y_n(s)+C_2(s)\frac{\nabla y_n(s)}{\nabla x(s)}+C_3(s)y_{n+m}(s)=0,\qquad m\in\mathbb{Z},\tag{3.18}$$

where $C_i(s)$, $i=1,2,3$, are polynomials coefficients. Notice that for the case $m=\pm1$ (3.18) becomes

$$C_1(s)y_n(s)+C_2(s)\frac{\nabla y_n(s)}{\nabla x(s)}+C_3(s)y_{n+1}(s)=0,\tag{3.19}$$

$$\widetilde{C}_1(s)y_n(s)+\widetilde{C}_2(s)\frac{\nabla y_n(s)}{\nabla x(s)}y_n(s)+\widetilde{C}_3(s)y_{n-1}(s)=0,\tag{3.20}$$

with polynomials coefficients $C_i(s)$ and $\widetilde{C}_i(s)$, $i=1,2,3$.

Remark 3.6. *Notice that all formulas obtained in this section are true for the linear lattice* $x(s)=s$. *In particular, the expressions (3.10), (3.14), (3.15), (3.16), (3.17), (3.18),(3.19) and (3.20) are valid. Notice also that for the linear lattice* $x(s)=s$ *and* $\Delta x(s)=1$.

3.1. **Applications to discrete systems.** As we already mentioned several important discrete systems can be described using the solutions of the difference equation of hypergeometric type (1.3). In fact, in several of these models the corresponding *wave functions* have the form

$$\psi(z) := \frac{\sqrt{\rho(z)}}{d_n} P_n(z),$$

where P_n is a classical discrete orthogonal polynomial of Hahn, Meixner, Kravchuk and Charlier, d_n is the norm of P_n and ρ the corresponding weight function.

To conclude this work we will show some results concerning the Charlier polynomials and the so-called Charlier oscillators (see e.g. [4, 10]). The main characteristics of the Charlier polynomials can be found in e.g. [1, 20].

The Charlier functions are defined by

$$\psi_n^\mu(z) = \sqrt{\frac{e^{-\mu}\mu^{z-n}}{\Gamma(z+1)n!}} C_n^\mu(z), \qquad n \geq 0.$$

Using the main data for the Charlier polynomials [1, 21] we obtain the following three term recurrence relation [5]

$$\sqrt{(n+1)\mu}\, \psi_{n+1}^\mu(z) + [(n+\mu) - z]\, \psi_n^\mu(z) + \sqrt{n\mu}\, \psi_{n-1}^\mu(z),$$

as well as the ladder type relations

$$\left[\sqrt{z}A_1 + \left(\sqrt{z} - \sqrt{\mu}\right) A_2\right] \psi_n^\mu(z) + A_2\sqrt{\mu}\nabla\psi_n^\mu(z) \\ + A_3\sqrt{z\mu^m(n+1)_m}\, \psi_{n+m}^\mu(z) = 0,$$

and

$$\left[\sqrt{\mu}B_1 + \left(\sqrt{z+1} - \sqrt{\mu}\right) B_2\right]\psi_n^\mu(z) + B_2\sqrt{z+1}\Delta\psi_n^\mu(z) \\ + B_3\sqrt{\mu^{m+1}(n+1)_m}\, \psi_{n+m}^\mu(z) = 0.$$

Let us remind here that $(a)_\nu$ denotes the Pochhammer symbol $(a)_\nu = \Gamma(a+\nu)/\Gamma(x)$.

In a similar way we can obtain several recurrence relations for the wave functions of the q-oscillators introduced e.g. in [2, 3, 8, 9].

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