RECURRENCE RELATIONS FOR DISCRETE HYPERGEOMETRIC FUNCTIONS

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Abstract. We present a general procedure for finding linear recurrence relations for the solutions of the second order difference equation of hypergeometric type. Applications to wave functions of certain discrete system are also given.

1. Introduction

In the last years there has been increasing interest in discrete models in classical and quantum physics (for a recent review see [20]). Several of such models are solved using the theory of the classical *discrete* polynomials [22]. Important instances of such systems are the discrete oscillators of Charlier [5], Kravchuk oscillators [6, 8, 10, 12, 14] and Meixner oscillators [5] that are related to the polynomials of Charlier, Kravchuk and Meixner, respectively, and the finite radial oscillator [9, 11] related with the Hahn polynomials. For applications it is important to have recurrence relations for the *discrete* wave function of such systems. Methods for obtaining such recurrence relations have attracted the interest of several authors (see e.g. [19, 20] and references therein).

Our main aim in this paper is to present a constructive approach for generating recurrence relations and ladder-type operators for some discrete system such as the discrete oscillators [2, 5, 6, 8, 9, 10, 11, 12, 13, 14], discrete Calogero-Sutherland model [20], etc. The main idea is to use the connection of the wave functions with the classical discrete polynomials in a similar way as it was done in our previous paper [16] for the N-th dimensional oscillators and hydrogenlike atoms. This approach allows us to recover the relations obtained in [5, 19, 20] and also to obtain several new relations for the discrete polynomials and therefore for the associated (wave) functions in a constructive way. This can be extended to other exactly solvable models which involve *discrete* hypergeometric functions or polynomials.

The structure of the paper is as follows: In section 2 the required results and notation from special function theory are introduced. The main results of the paper are in Section 3, where some general existence theorems are stated and proved. In Section 4 simple examples of recurrences and ladder-type relations of some discrete systems are presented. Finally, at the end of

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section 4, we include some *more complicate* examples, in order to show the interest as well as the power of the method for finding needed recurrence relations. In this way we show how this method can be useful for finding recurrences *à la carte*, that it seem to be, in general, a very important tool for computations with discrete systems.

2. The *discrete* ingredients

2.1. **"Discrete" preliminaries.**

Here we collect the basic background [1, 22] on hypergeometric discrete polynomials needed in the rest of the work.

Let us consider the second-order difference equation of hypergeometric-type

$$\sigma(s)\nabla\Delta y(s)+\tau(s)\Delta y(s)+\lambda y(s)=0\ , \tag{2.1}$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of degree not greater than 2 and 1, respectively, λ is a constant, and $\Delta f(s)=f(s+1)-f(s)$ and $\nabla f(s)=\Delta f(s-1)$ are the forward and backward difference operators, respectively. This equation can be written in self-adjoint form

$$\Delta[\sigma(s)\rho(s)\nabla y(s)]+\lambda\rho(s)y(s)=0, \tag{2.2}$$

where the function $\rho(s)$ satisfies the Pearson-type difference equation

$$\Delta\big[\sigma(s)\rho(s)\big]=\tau(s)\rho(s).$$

For the solutions of the difference equation (2.1) the following theorem holds

Theorem 2.1. [23, page 136] *The difference equation (2.1) has particular solutions of the form*

$$y_\nu(s)=\frac{C_\nu}{\rho(s)}\sum_{x=a}^{b-1}\frac{\rho_\nu(x)}{(x-s)_{\nu+1}} \tag{2.3}$$

if the condition

$$\left.\frac{\sigma(x)\rho_\nu(x)}{(x-s-1)_{\nu+2}}\right|_a^b=0, \tag{2.4}$$

is satisfied, and has solutions of the form

$$y_\nu(s)=\frac{C_\nu}{\rho(s)}\int_C\frac{\rho_\nu(x)\,dx}{(x-s)_{\nu+1}} \tag{2.5}$$

if the condition

$$\int_C\frac{\sigma(x+1)\rho(x+1)\,dx}{(x-s)_{\nu+2}}=\int_C\frac{\sigma(x)\rho(x)\,dx}{(x-s-1)_{\nu+2}} \tag{2.6}$$

is satisfied. Here C is a contour in the complex plane, C_ν is a constant, $\rho(s)$ and $\rho_\nu(s)$ are the solution of the Pearson type equations

$$\Delta\big[\sigma(s)\rho(s)\big]=\tau(s)\rho(s),\quad \Delta\big[\sigma(s)\rho_\nu(s)\big]=\tau(s)\rho_\nu(s), \tag{2.7}$$

where $\tau_\nu(s) = \sigma(s+\nu) - \sigma(s) + \tau(s+\nu)$, ν *is the root of the equation* $\lambda + \nu\tau' + 1/2\nu(\nu-1)\sigma'' = 0$, *and* $(x)_\nu$ *denotes the Pochhammer symbols or shifted factorials*

$$(x)_\nu := \frac{\Gamma(x+\nu)}{\Gamma(x)}. \tag{2.8}$$

Important instances of the functions y_ν are the classical discrete polynomials which are given by [23, page 139]

$$P_n(s) = \frac{n!B_n}{\rho(s)\ 2\pi i}\int_C \frac{\rho_n(x)}{(x-s)_{n+1}}dx, \tag{2.9}$$

when C is a closed contour surrounding the points $x = s, s-1, \ldots, s-n$ and it is assumed that $\rho_n(x)$ and $\rho_n(x+1)$ are analytic inside C, i.e., they correspond to formula (2.5). In this case

$$\lambda := \lambda_n = -n\Delta\tau(s) - \frac{1}{2}n(n-1)\sigma'', \quad n = 0, 1, 2, \ldots .$$

2.2. **The classical discrete polynomials.** The classical discrete orthogonal polynomials are orthogonal on the integers in $[a, b-1]$ with respect to the weight function $\rho(s)$, i.e.,

$$\sum_{x=a}^{b-1} P_n(s)P_m(s)\rho(s) = \delta_{nm}d_n^2,$$

provided that the boundary condition $\sigma(s)\rho(s)x^k\big|_{x=a,b} = 0$, for all $k \geq 0$, holds, where d_n^2 is the square of the norm of the polynomial $P_n(s)$. They can be obtained using the so-called Rodrigues-type formula

$$P_n(s) = \frac{B_n}{\rho(s)}\nabla^n[\rho_n(s)], \quad n = 0, 1, 2, \ldots\ ,\ .$$

where B_n is the normalization constant and

$$\rho_n(s) = \rho(s+n)\prod_{m=1}^{n}\sigma(s+m). \tag{2.10}$$

Furthermore, for the k-th differences we have [23, Eq. 20, page 110]

$$\Delta^k P_n(s) = \frac{A_{nk}B_n}{\rho_k(s)}\nabla^{n-k}[\rho_n(s)], \tag{2.11}$$

where

$$A_{nk} = \frac{n!}{(n-k)!}\prod_{m=0}^{k-1}\left[\tau' + (n+m-1)\frac{\sigma''}{2}\right]. \tag{2.12}$$

A simple consequence of the orthogonality is the three-term recurrence relation (TTRR) that the polynomials P_n satisfy

$$xP_n(s) = \alpha_n P_{n+1}(s) + \beta_n P_n(s) + \gamma_n P_{n-1}(s). \tag{2.13}$$

Also they satisfy several difference-recurrence relations [1]

$$\sigma(s)\nabla P_n(s) = \frac{\lambda_n}{n\tau_n'}\left[\tau_n(s)P_n(s) - \frac{B_n}{B_{n+1}}P_{n+1}(s)\right], \tag{2.14}$$

$$[\sigma(s)+\tau(s)]\Delta P_n(s) = \frac{\lambda_n}{n\tau_n'}\left\{[\tau_n(s) - n\tau_n']P_n(s) - \frac{B_n}{B_{n+1}}P_{n+1}(s)\right\}, \tag{2.15}$$

$$\sigma(s)\nabla P_n(s) = \widetilde{\alpha}_n P_{n+1}(s) + \widetilde{\beta}_n P_n(s) + \widetilde{\gamma}_n P_{n-1}(s), \tag{2.16}$$

$$[\sigma(s)+\tau(s)]\Delta P_n(s) = \widehat{\alpha}_n P_{n+1}(s) + \widehat{\beta}_n P_n(s) + \widehat{\gamma}_n P_{n-1}(s), \tag{2.17}$$

$$P_n(s) = Q_n(s) + \delta_n Q_{n-1}(s) + \epsilon_n Q_{n-2}(s), \tag{2.18}$$

where $Q_n(s) = \Delta P_{n+1}(s)/(n+1)$.

2.3. **Classical families of Hahn, Meixner, Kravchuk and Charlier.** The four families of classical discrete orthogonal polynomials are the Hahn $h_n^{\alpha,\beta}(x,N)$, Meixner $M_n^{\gamma,\mu}(s)$, Kravchuk $K_n^p(x,N)$ and Charlier $C_n^\mu(s)$, polynomials [18, 22, 23], whose main data in its monic form are shown in Tables 1, 2 and 3.

TABLE 1. Classification of discrete classical polynomials

P_n	Hahn $h_n^{\alpha,\beta}(s;N)$	Meixner $M_n^{\gamma,\mu}(s)$	Kravchuk $K_n^p(s)$	Charlier $C_n^\mu(s)$
$[a,b]$	$[0,N]$	$[0,\infty)$	$[0,N+1]$	$[0,\infty)$
σ	$s(N+\alpha-s)$	s	s	s
τ	$(\beta+1)(N-1)-(\alpha+\beta+2)s$	$(\mu-1)s+\mu\gamma$	$\dfrac{Np-s}{1-p}$	$\mu-s$
τ_n	$(\beta+1)(N-1)+n(N-\beta-n-2)$ $-(\alpha+\beta+2n+2)s$	$(\mu-1)s+\mu(\gamma+n)$	$\dfrac{(N-n)p-s}{1-p}$	$\mu-s$
$\sigma+\tau$	$(s+\beta+1)(N-1-s)$	$\mu s+\gamma\mu$	$-\dfrac{p}{1-p}(s-N)$	μ
λ_n	$n(n+\alpha+\beta+1)$	$(1-\mu)n$	$\frac{n}{1-p}$	n
ρ	$\dfrac{\Gamma(N+\alpha-s)\Gamma(\beta+s+1)}{\Gamma(N-s)\Gamma(s+1)}$ $\alpha,\beta\geq-1\ ,\ n\leq N-1$	$\dfrac{\mu^s\Gamma(\gamma+s)}{\Gamma(\gamma)\Gamma(s+1)}$ $\gamma>0, 0<\mu<1$	$\dfrac{N!p^s(1-p)^{N-s}}{\Gamma(N+1-s)\Gamma(s+1)}$ $0<p<1,\ n\leq N-1$	$\dfrac{e^{-\mu}\mu^s}{\Gamma(s+1)}$ $\mu>0$
ρ_n	$\dfrac{\Gamma(N+\alpha-s)\Gamma(n+\beta+s+1)}{\Gamma(N-n-s)\Gamma(s+1)}$	$\dfrac{\mu^{s+n}\Gamma(\gamma+n+s)}{\Gamma(\gamma)\Gamma(s+1)}$	$\dfrac{N!p^{s+n}(1-p)^{N-n-s}}{\Gamma(N+1-n-s)\Gamma(s+1)}$	$\dfrac{e^{-\mu}\mu^{s+n}}{\Gamma(s+1)}$

They can be expressed in terms of hypergeometric functions by [22, Section **2.7**,p. 49]:

$$h_n^{\alpha,\beta}(x,N) = \frac{(1-N)_n(\beta+1)_n}{(\alpha+\beta+n+1)_n}\,{}_3\mathrm{F}_2\left(\begin{array}{c} -x, \alpha+\beta+n+1, -n \\ 1-N, \beta+1 \end{array}\middle| 1\right),$$

$$M_n^{\gamma,\mu}(s) = (\gamma)_n\frac{\mu^n}{(\mu-1)^n}\quad {}_2\mathrm{F}_1\left(\begin{array}{c} -n, -x \\ \gamma \end{array}\middle| 1-\frac{1}{\mu}\right),$$

TABLE 2. Main data for monic Hahn and Chebyshev polynomials

	Hahn $h_n^{\alpha,\beta}(s;N)$	Chebyshev $t_n(s;N)=h_n^{0,0}(s;N)$
B_n	$\dfrac{(-1)^n}{(\alpha+\beta+n+1)_n}$	$\dfrac{(-1)^n}{(n+1)_n}$
b_n	$-\dfrac{n}{2}\left(\dfrac{2(\beta+1)(N-1)+(n-1)(\alpha-\beta+2N-2)}{\alpha+\beta+2n}\right)$	$-\dfrac{n(N-1)}{2}$
d_n^2	$\dfrac{n!\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+N+n+1)}{(\alpha+\beta+2n+1)(N-n-1)!\Gamma(\alpha+\beta+n+1)(\alpha+\beta+n+1)_n^2}$	$\dfrac{n!^2(N+n)!(n+1)_n^{-2}}{(2n+1)(N-n-1)!}$
β_n	$\dfrac{(\beta+1)(N-1)(\alpha+\beta)+n(2N+\alpha-\beta-2)(\alpha+\beta+n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}$	$\dfrac{N-1}{2}$
γ_n	$\dfrac{n(N-n)(\alpha+\beta+n)(\alpha+n)(\beta+n)(\alpha+\beta+N+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}$	$\dfrac{n^2(N^2-n^2)}{4(2n-1)(2n+1)}$
$\widehat{\alpha}_n$	$-n$	$-n$
$\widehat{\beta}_n$	$\dfrac{n(-\alpha-\beta-\alpha\beta-\beta^2-2n-2\alpha n-2\beta n-2n^2+\alpha N-\beta N)}{(\alpha+\beta+n+1)^{-1}(\alpha+\beta+2n)(\alpha+\beta+2n+2)}$	$-\dfrac{n(n+1)}{2}$
$\widehat{\gamma}_n$	$-\dfrac{n(\alpha+\beta+n)(\alpha+\beta+n+1)(\alpha+n)(\beta+n)(\alpha+\beta+N+n)}{(n-N)^{-1}(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}$	$-\dfrac{n^2(n+1)(N^2-n^2)}{4(2n-1)(2n+1)}$
$\widetilde{\alpha}_n$	$-n$	$-n$
$\widetilde{\beta}_n$	$\dfrac{n(\alpha+\alpha^2+\beta+\alpha\beta+2n+2\alpha n+2\beta n+2n^2+\alpha N-\beta N)}{(\alpha+\beta+n+1)^{-1}(\alpha+\beta+2n)(\alpha+\beta+2n+2)}$	$\dfrac{n(n+1)}{2}$
$\widetilde{\gamma}_n$	$-\dfrac{n(\alpha+\beta+n)(\alpha+\beta+n+1)(\alpha+n)(\beta+n)(\alpha+\beta+N+n)}{(n-N)^{-1}(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}$	$-\dfrac{n^2(n+1)(N^2-n^2)}{4(2n-1)(2n+1)}$
δ_n	$\dfrac{n(\alpha-\beta)(2N+\alpha+\beta)}{2(\alpha+\beta+2n)(\alpha+\beta+2n+2)}-\dfrac{n}{2}$	$-\dfrac{n}{2}$
ϵ_n	$\dfrac{n(n-1)(N-n)(\alpha+n)(\beta+n)(\alpha+\beta+N+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}$	$-\dfrac{n(n-1)(N^2-n^2)}{4(2n-1)(2n+1)}$

$$K_n^p(x,N)=\frac{(-p)^nN!}{(N-n)!}\;{}_2\mathrm{F}_1\left(\begin{array}{c}-n,-x\\-N\end{array}\middle|\frac{1}{p}\right),$$

$$C_n^\mu(s)=(-\mu)^n\;{}_2\mathrm{F}_0\left(\begin{array}{c}-n,-x\\-\end{array}\middle|-\frac{1}{\mu}\right),$$

where the generalized hypergeometric function ${}_p\mathrm{F}_q$ is defined by

$$ {}_p\mathrm{F}_q\left(\begin{array}{c}a_1,a_2,...,a_p\\b_1,b_2,...,b_q\end{array}\middle|x\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{x^k}{k!}. \tag{2.19}$$

A very important special case of the Hahn polynomials ($\alpha=\beta=0$) are the discrete Chebyshev polynomials $t_n(x,N):=h_n^{0,0}(x,N)$.

3. General recurrence relations

In this section we will obtain several recurrence relations for the solutions (2.3) and (2.5) of the difference equation (2.1). We start with the following lemma that is the discrete analog of Lemma in [23, page 14].

TABLE 3. Main data for monic Charlier, Meixner and Kravchuk polynomials

	Charlier $C_n^{\mu}(s)$	Kravchuk $K_n^{p}(s)$	Meixner $M_n^{\gamma,\mu}(s)$
B_n	$(-1)^n$	$(-1)^n(1-p)^n$	$\dfrac{1}{(\mu-1)^n}$
b_n	$-\dfrac{n}{2}(2\mu+n-1)$	$-n[Np+(n-1)(1/2-p)]$	$\left(\dfrac{n\mu}{\mu-1}\right)\left(\gamma+\dfrac{n-1}{2}\dfrac{\mu+1}{\mu}\right)$
d_n^2	$n!\mu^n$	$\dfrac{n!N!p^n(1-p)^n}{(N-n)!}$	$\dfrac{n!(\gamma)_n\mu^n}{(1-\mu)^{\gamma+2n}}$
β_n	$n+\mu$	$Np+(1-2p)n$	$\dfrac{n(1+\mu)+\mu\gamma}{1-\mu}$
γ_n	$n\mu$	$np(1-p)(N-n+1)$	$\dfrac{n\mu(n-1+\gamma)}{(\mu-1)^2}$
$\widehat{\alpha}_n$	0	0	0
$\widehat{\beta}_n$	0	$-\dfrac{np}{1-p}$	$n\mu$
$\widehat{\gamma}_n$	$n\mu$	$pn(N-n+1)$	$\dfrac{n\mu(n-1+\gamma)}{1-\mu}$
$\widetilde{\alpha}_n$	0	0	0
$\widetilde{\beta}_n$	n	n	n
$\widetilde{\gamma}_n$	$n\mu$	$pn(N-n+1)$	$\dfrac{n\mu(n-1+\gamma)}{1-\mu}$
δ_n	0	$n(1-p)$	$\dfrac{n\mu}{1-\mu}$
ϵ_n	0	0	0

Let us define the functions

$$\Phi_{\nu\mu}(z) = \sum_{s=a}^{b-1} \frac{\rho_\nu(s)}{(s-z)_{\mu+1}} \tag{3.1}$$

and

$$\Phi_{\nu\mu}(z) = \int_C \frac{\rho_\nu(s)ds}{(s-z)_{\mu+1}}, \tag{3.2}$$

corresponding to the functions[1] (2.3) and (2.5), respectively. In fact, the functions y_ν and the functions $\Phi_{\nu\mu}$ are related by the formula

$$y_\nu(z) = \frac{C_\nu}{\rho(z)}\Phi_{\nu\nu}(z). \tag{3.3}$$

Lemma 3.1. *Any three functions $\Phi_{\nu_i\mu_i}$, $i=1,2,3$ are connected by a linear relation*

$$\sum_{i=1}^{3} A_i(z)\Phi_{\nu_i\mu_i}(z) = 0, \tag{3.4}$$

[1]Obviously the functions (3.1) correspond to the boundary condition (2.4), whereas the functions (3.2) correspond to the condition (2.6).

with polynomial coefficients $A_i(z)$ which are not all three vanishing provided that the differences $\nu_i - \nu_j$ and $\mu_i - \mu_j$, $i, j = 1, 2, 3$ are integers and that the following condition holds[2]

$$\left.\frac{s^k\sigma(s)\rho_{\nu_0}(s)}{(s-z)_{\mu_0}}\right|_{s=a}^{s=b} = 0, \quad k = 0, 1, 2, \dots, \tag{3.5}$$

for the case when $\Phi_{\nu_i\mu_i}$ are given by (3.1) and

$$\int_C \Delta_s \frac{s^k\sigma(s)\rho_{\nu_0}(s)\,ds}{(s-z)_{\mu_0}} = 0, \quad k = 0, 1, 2, \dots, \tag{3.6}$$

for the case when $\Phi_{\nu_i\mu_i}$ are given by (3.2). Here ν_0 is the ν_i, $i = 1, 2, 3$ with the smallest real part and μ_0 is the μ_i, $i = 1, 2, 3$ with the largest real part.

Proof. We will give the proof for the case of functions of the form (3.1), the other case is completely similar. We have

$$\begin{aligned}
\sum_{i=1}^{3} A_i(z)\Phi_{\nu_i\mu_i}(z) &= \sum_{i=1}^{3} A_i(z) \sum_{s=a}^{b-1} \frac{\rho_{\nu_i}(s)}{(s-z)_{\mu_i+1}} = \sum_{s=a}^{b-1}\sum_{i=1}^{3} A_i(z)\frac{\rho_{\nu_i}(s)}{(s-z)_{\mu_i+1}} \\
&= \sum_{s=a}^{b-1} \frac{1}{(s-z)_{\mu_0+1}} \left(\sum_{i=1}^{3} A_i\rho_{\nu_i}(s)\frac{(s-z)_{\mu_0+1}}{(s-z)_{\mu_i+1}} \right) \\
&= \sum_{s=a}^{b-1} \frac{1}{(s-z)_{\mu_0+1}} \left(\sum_{i=1}^{3} A_i\rho_{\nu_i}(s)(s-z+\mu_i+1)_{\mu_0-\mu_i} \right),
\end{aligned}$$

where the identity

$$\frac{(s-z)_\alpha}{(s-z)_\beta} = (s-z+\beta)_{\alpha-\beta}, \qquad \alpha \ge \beta$$

is used. Next we use the identity $\rho_\nu(s) = \sigma(s+1)\rho_{\nu-1}(s+1)$ as well as the Pearson-type equation (2.7) rewritten in the equivalent form

$$\frac{\rho_\nu(s+1)}{\rho_\nu(s)} = \frac{\sigma(s)+\tau_\nu(s)}{\sigma(s+1)} = \frac{\sigma(s+\nu)+\tau(s+\nu)}{\sigma(s+1)},$$

which leads to

$$\frac{\rho(s+\nu_i)}{\rho(s+\nu_0)} = \frac{\sigma(s+\nu_i-1)+\tau(s+\nu_i-1)}{\sigma(s+\nu_i)} \cdots \frac{\sigma(s+\nu_0)+\tau(s+\nu_0)}{\sigma(s+\nu_0+1)}.$$

Thus, for all $\nu_i \ge \nu_0$

$$\rho_{\nu_i}(s) = [\sigma(s+\nu_0)+\tau(s+\nu_0)]\cdots[\sigma(s+\nu_i-1)+\tau(s+\nu_i-1)]\rho_{\nu_0}(s). \tag{3.7}$$

Using the last formula we obtain

$$\sum_{i=1}^{3} A_i(z)\Phi_{\nu_i\mu_i}(z) = \sum_{s=a}^{b-1} \frac{\rho_{\nu_0}(s)}{(s-z)_{\mu_0+1}}\,\Pi(s), \tag{3.8}$$

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[2]In some cases this condition is equivalent to the condition $s^k\sigma(s)\rho_{\nu_0}(s)|_{s=a}^{s=b} = 0$, $k = 0, 1, 2, \dots$.

where

$$\Pi(s) = \sum_{i=1}^{3} A_i(z)(s-z+\mu_i+1)_{\mu_0-\mu_i}\pi(s+\nu_0)\cdots\pi(s+\nu_i-1), \qquad (3.9)$$

is a polynomial in s and $\pi(s)=\sigma(s)+\tau(s)$. To conclude the proof we will show that the polynomials A_i, $i=1,2,3$ can be chosen such that

$$\frac{\rho_{\nu_0}(s)}{(s-z)_{\mu_0+1}}\,\Pi(s) = \Delta\left[\frac{\rho_{\nu_0+1}(s-1)}{(s-z)_{\mu_0}}\,Q(s)\right] = \Delta\left[\frac{\sigma(s)\rho_{\nu_0}(s)}{(s-z)_{\mu_0}}\,Q(s)\right], \quad (3.10)$$

where $Q(s)$ is a polynomial in s. Rewriting (3.8) with the help of the above formula and using the boundary condition (3.5) we find the expression[3]

$$\sum_{i=1}^{3} A_i(z)\Phi_{\nu_i\mu_i}(z) = 0.$$

Let us show that this polynomial Q and the polynomials A_i always exists. In fact, a straightforward computations give

$$\Delta\left[\frac{\sigma(s)\rho_{\nu_0}(s)}{(s-z)_{\mu_0}}\,Q(s)\right] = \frac{\rho_{\nu_0}(s)}{(s-z)_{\mu_0+1}}\times$$
$$\Bigg[\tau_{\nu_0}(s)(s-z)Q(s)+[\tau_{\nu_0}(s)+\sigma(s)](s-z)\Delta Q(s)-\mu_0\sigma(s)Q(s)\Bigg],$$

from which the following expression connecting the polynomials Π and Q follows

$$\Pi(s) = \Bigg[\tau_{\nu_0}(s)(s-z)-\mu_0\sigma(s)\Bigg]Q(s)+(s-z)[\tau_{\nu_0}(s)+\sigma(s)]\Delta Q(s). \quad (3.11)$$

From the above relation follows that the degree of $Q(s)$ is two less than the degree of $\Pi(s)$: it follows from the fact that the degree of τ_{ν_0} is less than or equal to 1. In fact, equating the coefficients of powers on the two sides of the above equation, we find a system of linear equations in the coefficients of $Q(s)$ and the coefficients A_i which has at least one unknown more than the number of equations. Notice that the coefficients of the unknowns are polynomials in z, so that after one coefficient is selected the remaining coefficients are rational functions of z, therefore after multiplying by the common denominator of the $A_i(z)$ we obtain the linear relation with polynomial coefficients. This completes the proof. □

Remark 3.2. *Notice that for the discrete polynomials (2.9) the condition (3.6) is automatically fulfilled, since the contour C is closed and ν is a non-negative integer, so Lemma 3.1 holds for any family of discrete polynomials of hypergeometric type. Notice also that Lemma (3.1) assures the existence of the non vanishing polynomials in (3.4) but does not give any method for*

[3] In the case of functions $\Phi_{\nu\mu}$ of the form (3.2) the proof is the same. In fact, changing the sums over s by the integral over C and using the boundary condition (3.6) we obtain the result.

finding them. Nevertheless, using (3.9) and (3.11), a constructive approach for finding the coefficients A_i, $i = 1, 2, 3$ *can explicitly be written down. This will be shown in a simple example connecting the* $\Phi_{\nu\,\nu-1}$, $\Phi_{\nu\,\nu}$, *and* $\Phi_{\nu\,\nu+1}$, *functions. In fact, for the case of classical polynomials of Jacobi, Laguerre, Hermite and Bessel, the corresponding Lemma* [23, page 14] *has been extensively used for deriving several recurrences relations and for getting the corresponding coefficients in a closed form in terms of the polynomials* σ *and* τ *(see e.g.* [17, 24], *and references therein). For the discrete case this study is under way.*

Example. Let us find the relation among $\Phi_{\nu\,\nu-1}$, $\Phi_{\nu\,\nu}$, and $\Phi_{\nu\,\nu+1}$. In this case since $\deg(\Pi) = 2$, $Q(z) = q_0$ is constant, and therefore (3.11) becomes

$$A_1(z)(s-z+\nu)_2 + A_2(z)(s-z+\nu+1) + A_3 = \Big[\tau_{\nu_0}(s)(s-z) - \mu_0\sigma(s)\Big]q_0.$$

Expanding both sides in powers of $s-z$ and comparing coefficients we obtain

$$A_1(z)\Phi_{\nu\,\nu-1}(z) + A_2(z)\Phi_{\nu\,\nu}(z) + A_3(z)\Phi_{\nu\,\nu+1}(z) = 0,$$

where

$$\begin{aligned} A_1(z) &= \tau' + (\nu-1)\frac{\sigma''}{2}, \quad A_2(z) = \tau(z) - \sigma'(z) + \nu\left(\tau' + \nu\frac{\sigma''}{2}\right), \\ A_3(z) &= -(\nu+1)\left[\sigma(z) + \nu\left(\tau' + (\nu-1)\frac{\sigma''}{2}\right)\right]. \end{aligned} \tag{3.12}$$

To conclude this section we write down the following two straightforward identities for the functions $\Phi_{\nu\mu}$

$$\Delta\Phi_{\nu\mu}(s) = (\mu+1)\Phi_{\nu\mu+1}(s+1), \tag{3.13}$$

and

$$\nabla\Phi_{\nu\mu}(s) = (\mu+1)\Phi_{\nu\mu+1}(s). \tag{3.14}$$

Recurrences involving the solutions y_ν. Let us now establish the following relevant relation

$$\Delta^k y_\nu(s) = \frac{C_\nu^{(k)}}{\rho_k(s)}\Phi_{\nu\,\nu-k}(s), \quad C_\nu^{(k)} = C_\nu \prod_{m=0}^{k-1}\left[\tau' + (\nu+m-1)\frac{\sigma''}{2}\right]. \tag{3.15}$$

This relation is valid for solutions of the form (2.3) and (2.5) of the difference equation (2.1). For the sake of simplicity we present here only the proof for the case of discrete polynomials. In this case (3.3) becomes

$$P_n(s) = \frac{C_n}{\rho(s)}\,\Phi_{n\,n}(s).$$

Comparing the last formula with the integral representation (2.9) we deduce that $C_n = B_n n!/(2\pi i)$.

Next, using the Cauchy integral

$$f(s) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-s}dz,$$

it follows that

$$\nabla^{n-k} f(s) = \frac{(n-k)!}{2\pi i} \int_C \frac{f(z)}{(z-s)_{n-k+1}} dz.$$

Then, using the Rodrigues formula (2.11) for the k-th differences $\Delta^k P_n(s)$ we obtain the integral representation

$$\Delta^k P_n(s) = \frac{A_{nk} B_n (n-k)!}{\rho_k(s)\; 2\pi i} \int_C \frac{\rho_n(z)}{(z-s)_{n-k+1}} dz = \frac{C_n (n-k)! A_{nk}}{n!\; \rho_k(s)} \Phi_{n\, n-k}(s),$$

from which, using (2.12), the relation (3.15) follows.

In the following $y_n^{(k)}(s)$ denotes the k-th differences $\Delta^k y_n(s)$.

Theorem 3.3. *Under the same conditions as in Lemma 3.1, any three functions* $y_{n_i}^{(k_i)}(s)$, $i = 1, 2, 3$ *are connected by a relation of the form*

$$\sum_{i=1}^{3} B_i(s) y_{n_i}^{(k_i)}(s) = 0, \tag{3.16}$$

with polynomials coefficients $B_i(s)$, $i = 1, 2, 3$.

Proof. From Lemma 3.1 we know that there exists three polynomials $A_i(s)$, $i = 1, 2, 3$ such that

$$\sum_{i=1}^{3} A_i(s) \Phi_{n_i, n_i - k_i}(s) = 0.$$

Then, using the relation (3.15) we find

$$\sum_{i=1}^{3} A_i(s) (C_\nu^{(k)})^{-1} \rho_{k_i}(s) y_{n_i}^{(k_i)}(s) = 0.$$

Now, dividing the last expression by $\rho_{k_0}(s)$, $k_0 = \min\{k_1, k_2, k_3\}$, and using (3.7) $\rho_{k_i}(s)/\rho_{k_0}(s) = \pi(s+k_0) \cdots \pi(s+k_i-1)$, we obtain

$$\sum_{i=1}^{3} B_i(s) y_{n_i}^{(k_i)}(s) = 0, \quad B_i(s) = A_i(s)(C_\nu^{(k)})^{-1} \pi(s+k_0) \cdots \pi(s+k_i-1),$$

which completes the proof. □

Corollary 3.4. *Under the same conditions as in Lemma 3.1, the following* Δ*-ladder-type relations hold*

$$B_1(s) y_n(s) + B_2(s) \Delta y_n(s) + B_3(s) y_{n+m}(s) = 0, \qquad m \in \mathbb{Z}, \tag{3.17}$$

with polynomials coefficients $B_i(s)$, $i = 1, 2, 3$.

Proof. It is sufficient to put $k_1 = k_3 = 0$, $k_2 = 1$, $n_1 = n_2 = n$ and $n_3 = n+m$ in (3.16). □

Notice that for the case $m=\pm1$ (3.17) becomes

$$B_1(s)y_n(s)+B_2(s)\Delta y_n(s)+B_3(s)y_{n+1}(s)=0, \tag{3.18}$$

$$\widetilde{B}_1(s)y_n(s)+\widetilde{B}_2(s)\Delta y_n(s)+\widetilde{B}_3(s)y_{n-1}(s)=0, \tag{3.19}$$

with polynomials coefficients $B_i(s)$ and $\widetilde{B}_i(s)$, $i=1,2,3$. The above relations (3.18) and (3.19) are usually called raising and lowering operators, respectively, for the functions y_n.

Let us now obtain raising and lowering operators for the functions y_ν but associated to the ∇ operators.

We start by applying the operator ∇ to the expression (3.3)

$$\nabla y_\nu(s)=C_\nu\Phi_{\nu\,\nu}(s)\left(\frac{1}{\rho(s)}-\frac{1}{\rho(s-1)}\right)+\frac{C_\nu}{\rho(s-1)}\nabla\Phi_{\nu\,\nu}(s),$$

or, equivalently,

$$\nabla\Phi_{\nu\,\nu}(s)=\frac{\rho(s-1)}{C_\nu}\nabla y_\nu(s)-\Phi_{\nu\,\nu}(s)\left(\frac{\rho(s-1)}{\rho(s)}-1\right).$$

Next we use (3.4) with $\nu_1=\mu_1=\nu_2=\nu$, $\mu_2=\nu+1$, and $\nu_3=\mu_3=\nu+m$ and substitute the above formula as well as (3.14) to obtain the relation

$$A_1(s)\Phi_{\nu\,\nu}(s)+\frac{A_2(s)}{\nu+1}\left[\frac{\rho(s-1)}{C_\nu}\nabla y_\nu(s)-\left(\frac{\rho(s-1)}{\rho(s)}-1\right)\Phi_{\nu\,\nu}(s)\right]$$
$$+A_3(s)\Phi_{\nu+m,\nu+m}(s)=0.$$

Finally, using the Pearson-type equation (2.7), $\rho(s-1)/\rho(s)=\sigma(s)/[\sigma(s-1)+\tau(s-1)]$, and multiplying the last expression by $C_\nu[\tau(s-1)+\sigma(s-1)]/\rho(s)$ yields

$$C_1(s)y_\nu(s)+C_2(s)\nabla y_\nu(s)+C_3(s)y_{\nu+m}(s)=0, \tag{3.20}$$

where

$$C_1(s)=(\nu+1)[\tau(s-1)+\sigma(s-1)]A_1(s)+[\sigma(s)-\tau(s-1)-\sigma(s-1)]A_2(s),$$

$$C_2(s)=\sigma(s)A_2(s),\quad C_3(s)=(\nu+1)[\sigma(s-1)+\tau(s-1)]C_\nu C_{\nu+m}^{-1}A_3(s),$$

i.e., we have proven the following

Theorem 3.5. *Under the same conditions as in Lemma 3.1, the functions $y_\nu(s)$ satisfy the ∇-ladder-type relations (3.20).*

If we now choose $m=\pm1$ we find the raising and lowering operators, respectively, for the functions y_n associated with the ∇-operator

$$C_1(s)y_n(s)+C_2(s)\nabla y_n(s)+C_3(s)y_{n+1}(s)=0, \tag{3.21}$$

$$\widetilde{C}_1(s)y_n(s)+\widetilde{C}_2(s)\nabla y_n(s)+\widetilde{C}_3(s)y_{n-1}(s)=0, \tag{3.22}$$

with polynomials coefficients $C_i(s)$ and $\widetilde{C}_i(s)$, $i=1,2,3$.

Remark 3.6. *Obviously the formulas (2.14)–(2.18) from section **2.2** are particular cases of the general expressions for the functions $y_\nu(s)$.*

4. Applications to discrete systems

As we already mentioned in the introduction, several important discrete systems (e.g. discrete oscillators) can be described using the solutions of the difference equation of hypergeometric type (2.1). In fact, in several of these models the corresponding *wave functions* have the form

$$\psi(z) := \frac{\sqrt{\rho(z)}}{d_n} P_n(z),$$

where P_n is a classical discrete orthogonal polynomial of Hahn, Meixner, Kravchuk and Charlier, d_n is the norm of P_n and ρ the corresponding weight function.

4.1. **Three-term recurrence relations.** Since P_n can be expressed by (2.9), we can use theorem 3.3 that assures the existence of the three-term recurrence relation

$$A_1 P_{n+1}(z) + A_2 P_n(z) + A_3 P_{n-1}(z) = 0. \tag{4.1}$$

But comparing it with (2.13) yields $A_1(z) = 1$, $A_2(z) = \beta_n - z$ and $A_3(z) = \gamma_n$, therefore, we may write

$$\frac{d_{n+1}}{d_n}\,\psi_{n+1}(z) + (\beta_n - z)\psi_n(z) + \gamma_n \frac{d_{n-1}}{d_n}\,\psi_{n-1}(z). \tag{4.2}$$

Charlier case. The Charlier functions are defined by

$$\psi_n^\mu(z) = \sqrt{\frac{e^{-\mu}\mu^{z-n}}{\Gamma(z+1)n!}}C_n^\mu(z), \qquad n \ge 0. \tag{4.3}$$

Using the main data for the Charlier polynomials (see Table 3) we obtain

$$\sqrt{(n+1)\mu}\,\psi_{n+1}^\mu(z) + [(n+\mu) - z]\,\psi_n^\mu(z) + \sqrt{n\mu}\,\psi_{n-1}^\mu(z). \tag{4.4}$$

Meixner case. In the Meixner case we have

$$\psi_n^{\gamma,\mu}(z) = \mu^{(z-n)/2}(1-\mu)^{\gamma/2+n}\sqrt{\frac{\Gamma(\gamma+z)}{\Gamma(\gamma)\Gamma(z+1)n!(\gamma)_n}}\,M_n^{\gamma,\mu}(z), \qquad n \ge 0, \tag{4.5}$$

thus

$$\begin{aligned}\sqrt{(n+1)\mu(\gamma+n)}\,\psi_{n+1}^{\gamma,\mu}(z) + &[n(1+\mu) + \mu\gamma - (1-\mu)z]\,\psi_n^{\gamma,\mu}(z)\\ &+ \sqrt{n\mu(\gamma+n-1)}\,\psi_{n-1}^{\gamma,\mu}(z).\end{aligned} \tag{4.6}$$

Kravchuk case. The Kravchuk functions defined by

$$\psi_n^p(z) = p^{(z-n)/2}(1-p)^{(N-n-z)/2}\sqrt{\frac{(N-n)!}{n!\Gamma(z+1)\Gamma(N-z+1)}}K_n^p(z,N), \tag{4.7}$$

are a finite set of orthogonal functions $n = 0, 1, \dots N$. For these functions (4.2) becomes

$$\begin{aligned}\sqrt{(n+1)p(1-p)(N-n)}\,\psi_{n+1}^p(z) + [Np + (1-2p)n - z]\,\psi_n^p(z)\\ + \sqrt{np(1-p)(N-n+1)}\,\psi_{n-1}^p(z) = 0.\end{aligned} \tag{4.8}$$

Hahn case. Finally, for the Hahn functions

$$\psi_n^{\alpha,\beta}(z) = \sqrt{\frac{\Gamma(N+\alpha-z)\Gamma(\beta+z+1)(\alpha+\beta+2n+1)\Gamma(N-n)\Gamma(\alpha+\beta+n+1)(\alpha+\beta+n+1)_n^2}{\Gamma(N-z)\Gamma(z+1)n!\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+N+n+1)}}\,h_n^{\alpha,\beta}(z) \tag{4.9}$$

we have

$$a_n\,\psi_{n+1}^{\alpha,\beta}(z) + (b_n - z)\psi_n^{\alpha,\beta}(z) + a_{n-1}\,\psi_{n-1}^{\alpha,\beta}(z) = 0, \tag{4.10}$$

where

$$a_n = \sqrt{\frac{n(N-n)(\alpha+n)(\beta+n)(\alpha+\beta+n)(\alpha+\beta+N+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}}$$

and

$$b_n = \frac{(\beta+1)(N-1)(\alpha+\beta) + n(2N+\alpha-\beta-2)(\alpha+\beta+n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}.$$

4.2. **Ladder-type Relations.** Let us look for relations involving the operators ∇ and Δ. Since $P_n(z) = d_n\psi_n(z)/\sqrt{\rho(z)}$, we have

$$\nabla P_n(z) = \frac{d_n}{\sqrt{\rho(z-1)}}\nabla\psi_n(z) + d_n\psi_n(z)\left(\frac{1}{\sqrt{\rho(z)}} - \frac{1}{\sqrt{\rho(z-1)}}\right). \tag{4.11}$$

On the other hand, Theorem 3.5 guarantees that for any integer m there exist polynomials A_1 , A_2 and A_3 which are not all three vanishing such that

$$A_1P_n(z) + A_2\nabla P_n(z) + A_3P_{n+m}(z) = 0\,. \tag{4.12}$$

Substituting (4.11) in (4.12) we get

$$\left[A_1 + A_2\left(1 - \sqrt{\frac{\rho(z)}{\rho(z-1)}}\right)\right]\psi_n(z) + A_2\sqrt{\frac{\rho(z)}{\rho(z-1)}}\nabla\psi_n(z) + A_3\frac{d_{n+m}}{d_n}\,\psi_{n+m}(z) = 0. \tag{4.13}$$

In a similar way, application of Δ to $P_n(z) = d_n\psi_n(z)/\sqrt{\rho(z)}$ gives

$$\Delta P_n(z) = \frac{d_n}{\sqrt{\rho(z+1)}}\Delta\psi_n(z) + d_n\left(\frac{1}{\sqrt{\rho(z+1)}} - \frac{1}{\sqrt{\rho(z)}}\right)\psi_n(z) \tag{4.14}$$

and consequently, substituting (4.14) into (3.17) gives

$$\left[B_1+B_2\left(\sqrt{\frac{\rho(z)}{\rho(z+1)}}-1\right)\right]\psi_n(z)+B_2\sqrt{\frac{\rho(z)}{\rho(z+1)}}\Delta\psi_n(z) \qquad +B_3\frac{d_{n+m}}{d_n}\,\psi_{n+m}(z)=0 \tag{4.15}$$

Let us point out that for getting the polynomial coefficients B_i, $i=1,2,3$ in (3.17) (or A_i, $i=1,2,3$ in (4.12)) we will follow the idea in [16]. Given m, the idea is to identify the corresponding relation (3.17) (or (4.12)) with one of the known expressions (2.13)–(2.18) for the classical polynomials. In doing so, after m is choosen and fixed, (3.17) (or (4.12)) may or may not transform into one of the known formulas (2.13)–(2.18). In the first situation, which is the simplest one, we can identify directly the coefficients B_i, $i=1,2,3$ (or A_i, $i=1,2,3$) comparing the formula (3.17) (or (4.12)) with one of the known expressions (2.13)–(2.18). In the second one, in general we need to combine (3.17) (or (4.12)) with a certain combination of two or more formulas (2.13)–(2.18) to obtain the unknown polynomials B_i, $i=1,2,3$ (or A_i, $i=1,2,3$). In the following examples we show how this works in the first situation. The more "complex" cases will be considered in the next subsection 4.3.2.

Charlier case. Substituting (4.3) into (4.13) and (4.15), gives, respectively

$$\left[\sqrt{z}A_1+\left(\sqrt{z}-\sqrt{\mu}\right)A_2\right]\psi_n^\mu(z)+A_2\sqrt{\mu}\nabla\psi_n^\mu(z) \qquad +A_3\sqrt{z\mu^m(n+1)_m}\,\psi_{n+m}^\mu(z)=0, \tag{4.16}$$

$$\left[\sqrt{\mu}B_1+\left(\sqrt{z+1}-\sqrt{\mu}\right)B_2\right]\psi_n^\mu(z)+B_2\sqrt{z+1}\Delta\psi_n^\mu(z) \qquad +B_3\sqrt{\mu^{m+1}(n+1)_m}\,\psi_{n+m}^\mu(z)=0. \tag{4.17}$$

Let us remind here that $(a)_k$ denotes the Pochhammer symbol (2.8).

Now we proceed by choosing particular values for the parameter $m\in\mathbb{Z}$.

• We start with the case $m=-1$. Then (4.12) becomes

$$A_1C_n^\mu(z)+A_2\nabla C_n^\mu(z)+A_3C_{n-1}^\mu(z)=0\,.$$

Comparing the above equation with (2.16) we find $A_1=-n$, $A_2=z$, $A_3=-n\mu$, and therefore (4.16) becomes

$$\left(z-\sqrt{\mu z}-n\right)\psi_n^\mu(z)+\sqrt{\mu z}\nabla\psi_n^\mu(z)-\sqrt{n\mu}\,\psi_{n-1}^\mu(z)=0. \tag{4.18}$$

On the other hand, (3.17) becomes

$$B_1C_n^\mu(z)+B_2\Delta C_n^\mu(z)+B_3C_{n-1}^\mu(z)=0\,,$$

so, comparing with (2.17) gives $B_1=0$, $B_2=1$, $B_3=-n$. Therefore, (4.17) gives

$$\left(\sqrt{z+1}-\sqrt{\mu}\right)\psi_n^\mu(z)+\sqrt{z+1}\Delta\psi_n^\mu(z)-\sqrt{n}\,\psi_{n-1}^\mu(z)=0. \tag{4.19}$$

• Analogously, for the case $m = 1$, (4.12) becomes

$$A_1 C_n^{\mu}(z) + A_2 \nabla C_n^{\mu}(z) + A_3 C_{n+1}^{\mu}(z) = 0 \, .$$

Now, comparing with (2.14) gives $A_1 = \mu - z$, $A_2 = z$, $A_3 = 1$, thus, (4.16) becomes

$$\left(\sqrt{\mu} - \sqrt{z}\right) \psi_n^{\mu}(z) + \sqrt{z} \nabla \psi_n^{\mu}(z) + \sqrt{n+1}\, \psi_{n+1}^{\mu}(z) = 0, \tag{4.20}$$

In this case (3.17) has the form

$$B_1 C_n^{\mu}(z) + B_2 \Delta C_n^{\mu}(z) + B_3 C_{n+1}^{\mu}(z) = 0 \, .$$

If we compare the above expression with (2.15) we find $B_1 = \mu + n - z$, $B_2 = \mu$, $B_3 = 1$, so (4.17) transforms to

$$\begin{aligned} \big[(\mu - z)(\mu + n - z) - \mu(\sqrt{\mu} - \sqrt{z+1})\big] \psi_n^{\mu}(z) + \mu\sqrt{z+1} \Delta \psi_n^{\mu}(z) \\ + \mu\sqrt{n+1}\, \psi_{n+1}^{\mu}(z) = 0. \end{aligned} \tag{4.21}$$

Meixner case. Substituting (4.5) into (4.13) and (4.15), we get, respectively

$$\begin{aligned} \left[\sqrt{z} A_1 + A_2 \left(\sqrt{z} - \sqrt{\mu(\gamma - 1 + z)}\right)\right] \psi_n^{\gamma,\mu}(z) + A_2 \sqrt{\mu(\gamma - 1 + z)} \nabla \psi_n^{\gamma,\mu}(z) \\ + A_3 \sqrt{\frac{z \mu^m (n+1)_m (\gamma + n)_m}{(1-\mu)^{2m}}}\, \psi_{n+m}^{\gamma,\mu}(z) = 0, \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} \left[\sqrt{\mu(\gamma + z)} B_1 + B_2 \left(\sqrt{z+1} - \sqrt{\mu(\gamma + z)}\right)\right] \psi_n^{\gamma,\mu}(z) + B_2 \sqrt{z+1} \Delta \psi_n^{\gamma,\mu}(z) \\ + B_3 \sqrt{\frac{(\gamma + z) \mu^{m+1} (n+1)_m (\gamma + n)_m}{(1-\mu)^{2m}}} \psi_{n+m}^{\gamma,\mu}(z) = 0. \end{aligned} \tag{4.23}$$

Now we proceed by choosing particular values for the parameter $m \in \mathbb{Z}$.

• For $m = -1$ (4.12) becomes

$$A_1 M_n^{\gamma,\mu}(z) + A_2 \nabla M_n^{\gamma,\mu}(z) + A_3 M_{n-1}^{\gamma,\mu}(z) = 0 \, .$$

Comparing with (2.16) we get $A_1 = n$, $A_2 = -z$, $A_3 = \frac{n\mu(n-1+\gamma)}{1-\gamma}$, so (4.22) becomes

$$\begin{aligned} &\left[n\sqrt{z} - z\left(\sqrt{z} - \sqrt{\mu(\gamma - 1 + z)}\right)\right] \psi_n^{\gamma,\mu}(z) \\ &- z\sqrt{\mu(\gamma - 1 + z)} \nabla \psi_n^{\gamma,\mu}(z) + \sqrt{z n \mu (n - 1 + \gamma)} \psi_{n-1}^{\gamma,\mu}(z) = 0. \end{aligned} \tag{4.24}$$

In this case (3.17) has the form

$$B_1 M_n^{\gamma,\mu}(z) + B_2 \Delta M_n^{\gamma,\mu}(z) + B_3 M_{n-1}^{\gamma,\mu}(z) = 0 \, .$$

Comparing the last equation with (2.17) yields $B_1 = n$, $B_2 = -z - \gamma$, and $B_3 = \frac{n(n-1+\gamma)}{1-\mu}$, therefore (4.23) gives

$$\begin{aligned}&\left[n\sqrt{\mu} - \sqrt{z+\gamma}\left(\sqrt{z+1} - \sqrt{\mu(\gamma+z)}\right)\right]\psi_n^{\gamma,\mu}(z)\\ &- \sqrt{(z+1)(z+\gamma)}\Delta\psi_n^{\gamma,\mu}(z) + \sqrt{n(n-1+\gamma)}\,\psi_{n-1}^{\gamma,\mu}(z) = 0.\end{aligned} \tag{4.25}$$

• Choosing $m = 1$, (4.12) becomes

$$A_1 M_n^{\gamma,\mu}(z) + A_2 \nabla M_n^{\gamma,\mu}(z) + A_3 M_{n+1}^{\gamma,\mu}(z) = 0\ .$$

A solution now is (compare with (2.14))

$$A_1 = \mu(\gamma+n) - (1-\mu)z, \quad A_2 = z, \quad A_3 = 1-\mu,$$

thus (4.22) becomes

$$\begin{aligned}&\left[\left(\mu(\gamma+n+z) - \sqrt{\mu z(\gamma-1+z)}\right)\right]\psi_n^{\gamma,\mu}(z)\\ &+\sqrt{\mu z(\gamma-1+z)}\nabla\psi_n^{\gamma,\mu}(z) + \sqrt{\mu(n+1)(\gamma+n)}\,\psi_{n+1}^{\gamma,\mu}(z) = 0.\end{aligned} \tag{4.26}$$

In this case (3.17) takes the form

$$B_1 M_n^{\gamma,\mu}(z) + B_2 \Delta M_n^{\gamma,\mu}(z) + B_3 M_{n+1}^{\gamma,\mu}(z) = 0\ ,$$

which comparing with (2.15) gives $B_1 = \mu(z+\gamma) + n - z$, $B_2 = \mu(z+\gamma)$, and $B_3 = 1-\mu$. Thus (4.23) gives

$$\begin{aligned}&\left[\sqrt{\mu^{-1}}\Big(\mu(\gamma+z) + n - z\Big) + \sqrt{z+\gamma}\left(\sqrt{z+1} - \sqrt{\mu(\gamma+z)}\right)\right]\psi_n^{\gamma,\mu}(z)\\ &+ \sqrt{(z+1)(z+\gamma)}\Delta\psi_n^{\gamma,\mu}(z) + \sqrt{(n+1)(\gamma+n)}\,\psi_{n+1}^{\gamma,\mu}(z) = 0.\end{aligned} \tag{4.27}$$

Kravchuk case. In this case using (4.7), (4.13) and (4.15) we have

$$\begin{aligned}&\left[\sqrt{(1-p)z}A_1 + A_2\left(\sqrt{(1-p)z} - \sqrt{p(N+1-z)}\right)\right]\psi_n^p(z)\\ &\qquad +A_2\sqrt{p(N+1-z)}\nabla\psi_n^p(z)\\ &+A_3\sqrt{z(n+1)_m(N-n-m+1)_m p^m(1-p)^{m+1}}\,\psi_{n+m}^p(z) = 0,\end{aligned} \tag{4.28}$$

$$\begin{aligned}&\left[\sqrt{p(N-z)}A_1 + A_2\left(\sqrt{(1-p)(z+1)} - \sqrt{p(N-z)}\right)\right]\psi_n^p(z)\\ &\qquad +A_3\sqrt{(1-p)(z+1)}\Delta\psi_n^p(z)\\ &+A_2\sqrt{(N-z)(n+1)_m(N-n-m+1)_m p^{m+1}(1-p)^m}\,\psi_{n+m}^p(z) = 0.\end{aligned} \tag{4.29}$$

• For $m = -1$, (4.12) becomes

$$A_1 K_n^p(z) + A_2 \nabla K_n^p(z) + A_3 K_{n-1}^p(z) = 0\ ,$$

for which the solution is (see (2.16))

$$A_1 = n, \quad A_2 = -z, \quad A_3 = pn(N-n+1),$$

and therefore (4.28) becomes

$$\begin{aligned}&\left[\left(\sqrt{(1-p)z}-\sqrt{p(N+1-z)}\right)z-n\sqrt{(1-p)z}\right]\psi_n^p(z)\\&+\sqrt{p(N+1-z)}z\nabla\psi_n^p(z)-\sqrt{zpn(N-n+1)}\,\psi_{n-1}^p(z)=0.\end{aligned}\tag{4.30}$$

Now (3.17) gives

$$B_1K_n^p(z)B_2\Delta K_n^p(z)+B_3K_{n-1}^p(z)=0\,,$$

from which, comparing with (2.17), we find $B_1=n$, $B_2=N-z$, and $B_3=-n(N-n+1)(1-p)$. Then (4.29) yields

$$\begin{aligned}&\left[n\sqrt{p}+\sqrt{N-z}\left(\sqrt{(1-p)(z+1)}-\sqrt{p(N-z)}\right)\right]\psi_n^p(z)\\&+\sqrt{(1-p)(z+1)(N-z)}\Delta\psi_n^p(z)-\sqrt{n(N-n+1)(1-p)}\,\psi_{n-1}^p(z)=0.\end{aligned}\tag{4.31}$$

• Now, for $m=1$, (4.12) has the form

$$A_1K_n^p(z)+A_2\nabla K_n^p(z)+A_3K_{n+1}^p(z)=0$$

for which $A_1=(N-n)p-z$, $A_2=(1-p)z$, $A_3=1$, (see (2.14)) so (4.28) becomes

$$\begin{aligned}&\left[\sqrt{z}\frac{(N-n)p-z}{\sqrt{p(1-p)}}+\left(\sqrt{\frac{1-p}{p}}z-\sqrt{N+1-z}\right)z\right]\psi_n^p(z)\\&\qquad+z\sqrt{N+1-z}\nabla\psi_n^p(z)+\sqrt{(n+1)(N-n)z}\,\psi_{n+1}^p(z)=0.\end{aligned}\tag{4.32}$$

Since for this case (3.17) has the form

$$B_1K_n^p(z)+B_2\Delta K_n^p(z)+B_3K_{n+1}^p(z)=0$$

then, comparing with (2.15), $B_1=(N-n)p+n-z$, $B_2=p(N-z)$, and $B_3=1$, so (4.29) transforms into

$$\begin{aligned}&\left\{\frac{\sqrt{p(N-z)}}{1-p}\left[(N-n)p+n-z\right]+\left[\sqrt{(1-p)(z+1)}-\sqrt{p(N-z)}\right]z\right\}\psi_n^p(z)\\&+z\sqrt{(1-p)(z+1)}\Delta\psi_n^p(z)+p\sqrt{(1-p)(n+1)(N-n)(N-z)}\,\psi_{n+1}^p(z)=0.\end{aligned}\tag{4.33}$$

Hahn case. As in the previous cases we substitute (4.9) in (4.13) and (4.15) to obtain, respectively, the expressions

$$\begin{aligned}&\left[A_1\sqrt{\sigma(z)}+A_2g(z)\right]\psi_n^{\alpha,\beta}(z)+A_2\sqrt{\pi(z-1)}\nabla\psi_n^{\alpha,\beta}(z)\\&\qquad+A_3\sqrt{\sigma(z)}\,\frac{d_{n+m}}{d_n}\,\psi_{n+m}^{\alpha,\beta}(z)=0,\end{aligned}\tag{4.34}$$

and

$$\begin{aligned}\Big[B_1\sqrt{\pi(z)}+B_2 g(z+1)\Big]\,\psi_n^{\alpha,\beta}(z)+B_2\sqrt{\sigma(z+1)}\Delta\psi_n^{\alpha,\beta}(z)&\\ +B_3\sqrt{\pi(z)}\,\frac{d_{n+m}}{d_n}\,\psi_{n+m}^{\alpha,\beta}(z)=0,&\end{aligned}\tag{4.35}$$

where $\sigma(z)=z(N+\alpha-z)$, $\pi(z)=(z+\beta+1)(N-z-1)$, and $g(z)=\sqrt{\sigma(z)}-\sqrt{\pi(z-1)}$. In the following we will use also the following notation

$$f(z)=\sqrt{\sigma(z)\pi(z-1)}=\sqrt{z(N+\alpha-z)(z+\beta)(N-z)},$$

and

$$\theta_n=\sqrt{\frac{n(N-n)(\alpha+n)(\beta+n)(\alpha+\beta+n)(\alpha+\beta+N+n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)}}.$$

Using the same technique as in the previous cases we arrive at the following expressions

- $m=-1$

$$\Big[n(\epsilon_n-z)-\sqrt{\sigma(z)}g(z)\Big]\,\psi_n^{\alpha,\beta}(z)-f(z)\nabla\psi_n^{\alpha,\beta}(z)+(\alpha+\beta+2n+1)\theta_n\psi_{n-1}^{\alpha,\beta}(z)=0,\tag{4.36}$$

where $\epsilon_n=\frac{(\beta+1)(N-1)(\alpha+\beta)+(\alpha+\beta+n+1)[(\alpha+\beta)(\alpha+1+2n)+(n+N)(\alpha-\beta+2n)]}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}$ and

$$\begin{aligned}\Big[n(z+\varsigma_n)+\sqrt{\pi(z)}g(z+1)\Big]\psi_n^{\alpha,\beta}(z)+f(z+1)\Delta\psi_n^{\alpha,\beta}(z)&\\ -(\alpha+\beta+2n+1)\theta_n\,\psi_{n-1}^{\alpha,\beta}(z)=0,&\end{aligned}\tag{4.37}$$

where $\varsigma_n=\alpha+\beta+n+1-\frac{(\alpha+\beta)(\alpha+\beta+N+n)}{\alpha+\beta+2n}$.

- $m=1$

$$\begin{aligned}\Big[(\alpha+\beta+n+1)(\varepsilon_n-z)-\sqrt{\sigma(z)}g(z)\Big]\psi_n^{\alpha,\beta}(z)-f(z)\nabla\psi_n^{\alpha,\beta}(z)&\\ +(\alpha+\beta+1)\theta_{n+1}\psi_{n+1}^{\alpha,\beta}(z)=0,&\end{aligned}\tag{4.38}$$

where

$$\varepsilon_n=\frac{(\alpha+\beta)[n(\alpha+1+2n)+(N-1)(\beta+1+2n)]+n(\alpha-\beta)(\alpha+\beta+n+N+1)+2n(n+1)(n+N-1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)},$$

and

$$\begin{aligned}\Big[(\alpha+\beta+1)\,(\xi_n-z)+\sqrt{\pi(z)}g(z+1)\Big]\psi_n^{\alpha,\beta}(z)+f(z+1)\Delta\psi_n^{\alpha,\beta}(z)&\\ +(\alpha+\beta+2n+1)\theta_{n+1}\,\psi_{n+1}^{\alpha,\beta}(z)=0,&\end{aligned}\tag{4.39}$$

where $\xi_n=n+\frac{(\beta+1)(N-1)+n(N-\beta-n-2)}{\alpha+\beta+2n+2}$.

4.3. **Further examples.** To conclude this work let us show how we can use the method presented here for finding some higher recurrence relations for the Charlier polynomials. Notice that in the following four examples one should use a combination of known relations of discrete orthogonal polynomials. In a similar way, we can proceed with the other families to obtain more recurrence relations.

4.3.1. *Recurrences for Charlier polynomials.* We start with some examples of the recurrences (3.16) connecting Charlier polynomials.

Choosing, in (3.16),

$$n_1 = n-1\ ,\ \ n_2 = n\ ,\ \ n_3 = n+1\ \ ;\quad k_1 = 1\ ,\ \ k_2 = 1\ ,\ \ k_3 = 0$$

we find

$$B_1(z)\,\Delta C^{\mu}_{n-1}(z) + B_2(z)\,\Delta C^{\mu}_{n}(z) + B_3(z) C^{\mu}_{n+1}(z) = 0\,.$$

Hence, by (2.17), $\Delta C^{\mu}_n(z) = nC^{\mu}_{n-1}(z)$, it becomes into

$$(n-1)B_1(z)C^{\mu}_{n-2}(z) + nB_2(z)C^{\mu}_{n-1}(z) + B_3(z)C^{\mu}_{n+1}(z) = 0\,.$$

Using, now, the TTRR for the Charlier polynomials (cf. (2.13)), the last relation transforms into

$$(n-1)B_1(z)C^{\mu}_{n-2}(z) + n\left[B_2(z) - \mu B_3(z)\right]C^{\mu}_{n-1}(z) + (z-n-\mu)B_3(z)C^{\mu}_n(z) = 0.$$

Therefore, comparing with (2.13), we obtain

$$B_1(z) = n\mu(z-n-\mu)\ ,\ \ B_2(z) = n\mu - (z-n-\mu)(z-n+1-\mu)\ ,\ \ B_3(z) = n\,,$$

which leads to

$$nC^{\mu}_{n+1}(z) = \left[(z-n-\mu)(z-n+1-\mu) - n\mu\right]\Delta C^{\mu}_n(z) - n\mu(z-n-\mu)\,\Delta C^{\mu}_{n-1}(z).$$

Let us now choose,

$$n_1 = n-1\ ,\ \ n_2 = n\ ,\ \ n_3 = n+3\ \ ;\quad k_1 = 0\ ,\ \ k_2 = 0\ ,\ \ k_3 = 2.$$

Then (3.16) reads

$$B_1(z)C^{\mu}_{n-1}(z) + B_2(z)C^{\mu}_n(z) + B_3(z)\,\Delta^2 C^{\mu}_{n+3}(z) = 0\,.$$

Using now (2.17), it becomes into

$$B_1(z)C^{\mu}_{n-1}(z) + B_2(z)C^{\mu}_n(z) + (n+3)(n+2)B_3(z)C^{\mu}_{n+1}(z) = 0\,.$$

If we compare the last expression with the TTRR for Charlier polynomials (cf. (2.13)) one finds

$$B_1(z) = n(n+2)(n+3)\mu\ ,\ \ B_2(z) = (n+2)(n+3)(n+\mu-z)\ ,\ \ B_3(z) = 1\,.$$

Then,

$$\Delta^2 C^{\mu}_{n+3}(z) = (n+2)(n+3)(z-n-\mu)C^{\mu}_n(z) - n(n+2)(n+3)\mu C^{\mu}_{n-1}(z)\,.$$

4.3.2. *Higher order ladder-type relations for Charlier functions ψ_n^μ.* Let us now chose $m=-2$ in (4.16). Then (4.12) becomes

$$A_1 C_n^\mu(z) + A_2 \nabla C_n^\mu(z) + A_3 C_{n-2}^\mu(z) = 0\,, \tag{4.40}$$

which does not correspond to any of the known relations for the Charlier polynomials. To obtain the coefficients A_1, A_2 and A_3, we can proceed as follows: Using the TTRR (cf. (2.13)) for the Charlier polynomials

$$zC_{n-1}^\mu(z) = C_n^\mu(z) + (n-1+\mu)C_{n-1}^\mu(z) + (n-1)\mu C_{n-2}^\mu(z),$$

(4.40) transforms into

$$\left(A_1 - \frac{A_3}{(n-1)\mu}\right) C_n^\mu(z) + A_2 \nabla C_n^\mu(z) + A_3 \frac{x-n+1-\mu}{\mu(n-1)} C_{n-1}^\mu(z) = 0\,.$$

Now comparing with (2.16) we obtain

$$A_1(z) = -\frac{n(z-n+1)}{z-n+1-\mu}, \quad A_2(z) = z, \quad A_3 = -\frac{n(n-1)\mu^2}{z-n+1-\mu}.$$

Now, multiplying by $z-n+1-\mu$ we finally obtain

$$-n(z-n+1)C_n^\mu(z) + z(z-n+1-\mu)\nabla C_n^\mu(z) - n(n-1)\mu^2 C_{n-2}^\mu(z) = 0\,.$$

Thus, we find for the functions $\psi_n^\mu(z)$ the relation

$$\begin{aligned} 0 = &\left[-n(z-n+1)\sqrt{z} + \left(\sqrt{z}-\sqrt{\mu}\right) z(z-\mu-n+1)\right] \psi_n^\mu(z) \\ &+ \sqrt{\mu} z(z-\mu-n+1)\nabla\psi_n^\mu(z) - \mu\sqrt{zn(n-1)}\,\psi_{n-2}^\mu(z). \end{aligned} \tag{4.41}$$

Similarly, substituting $m=2$ in (3.17) we find

$$B_1(z)C_n^\mu(z) + B_2(z)\,\Delta C_n^\mu(z) + B_3(z) C_{n+2}^\mu(z) = 0\,.$$

Now, using twice the TTRR (cf. (2.13))

$$C_{n+1}^\mu(z) = (z-n-\mu)C_n^\mu(z) - nC_{n-1}^\mu(z)\,,$$

we obtain

$$\begin{aligned} &B_2(z)\Delta C_n^\mu(z) = n\mu(z-n-1-\mu)B_3(z)C_{n-1}^\mu(z) \\ &- \{B_1(z) + [(z-n-1-\mu)(z-n-\mu)-(n+1)\mu]\,B_3(z)\}\,C_n^\mu(z) \end{aligned} \tag{4.42}$$

Comparing (4.42) with (2.17) one finds

$$B_1(z) = (n+1)\mu - (z-n-1-\mu)(z-n-\mu)\,, \quad B_2(z) = \mu(z-n-1-\mu)$$

and $B_3(z)=1$. This leads to the following relation for Charlier functions ψ_n^μ

$$\begin{aligned} &\Big\{\sqrt{\mu}\,[(n+1)\mu - (z-n-1-\mu)(z-n-\mu)] + \\ &\qquad\qquad \left(\sqrt{z+1}-\sqrt{\mu}\right)\mu(z-n-1-\mu)\Big\}\psi_n^\mu(z) \\ &+ \mu(z-n-1-\mu)\sqrt{z+1}\,\Delta\psi_n^\mu(z) - \sqrt{\mu^3(n+1)(n+2)}\psi_{n+2}^\mu(z) = 0\,, \end{aligned} \tag{4.43}$$

that corresponds to the ladder-type relation (4.17) with $m=2$.

Concluding remarks. In this paper we present a simple, unified and constructive approach for finding linear recurrence relations for the difference hypergeometric-type functions, i.e., solutions of the hypergeometric difference equation (2.1), and apply the general results to some discrete models (e.g. discrete oscillators). Furthermore, the method described here is valuable for new situations, such as higher order recurrence relations and ladder-type relations for the classical discrete orthogonal polynomials.

Other important instances of discrete systems are the so-called q-oscillators (see e.g. [3, 6, 7, 15, 21] and reference therein) that are related with the q-polynomials. For these cases only a few recurrences are known [4]. A more detailed study of these q-models is under way.

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