

Orthogonal matrix polynomials satisfying second order difference equations *

R. Álvarez-Nodarse, A. J. Durán and A. M. de los Ríos

Departamento de Análisis Matemático. Universidad de Sevilla

Apdo (P. O. BOX) 1160. 41080 Sevilla. Spain

ran@us.es, duran@us.es, amdelosrios@us.es

Abstract

We develop a method that allows us to construct families of orthogonal matrix polynomials of size $N \times N$ satisfying second order difference equations with polynomial coefficients. The existence (and properties) of these orthogonal families strongly depends on the non commutativity of the matrix product, the existence of singular matrices and the matrix size N .

1 Introduction and results

The theory of matrix valued orthogonal polynomials starts with two papers by M. G. Kreĭn in 1949, see [15, 16]. The orthogonality is with respect to a weight matrix W , that is, a $N \times N$ matrix of measures supported in the real line such that $W(A)$ is positive semidefinite for any Borel set $A \subset \mathbb{R}$, having finite moments and satisfying that $\int P(x)dW(x)P^*(x)$ is nonsingular if the leading coefficient of the matrix polynomial P is nonsingular. We associate to a weight matrix W a Hermitian sesquilinear form defined by

$$(1) \quad \langle P, Q \rangle = \int P(x)dW(x)Q^*(x)$$

where P and Q are matrix polynomials. We then say that a sequence of matrix polynomials $(P_n)_n$, P_n of degree n with nonsingular leading coefficient, is orthogonal with respect to W if $\langle P_n, P_k \rangle = \Lambda_n \delta_{k,n}$, where Λ_n is a positive definite matrix for $n \geq 0$.

Nevertheless, until recently this theory suffered from a lack of interesting examples. The situation has dramatically changed in the last eight years, where

*Partially supported by MTM2009-12740-C03-02 (Ministerio de Economía y Competitividad), FQM-262, FQM-4643, FQM-7276 (Junta de Andalucía) and Feder Funds (European Union).

one can see a growing number of papers devoted to introduce and study many families of orthogonal matrix polynomials satisfying second (and higher) order differential equations of the form

$$(2) \quad P_n''(x)F_2(x) + P_n'(x)F_1(x) + P_n(x)F_0 = \Lambda_n P_n(x), \quad n = 0, 1, \dots$$

where F_2 , F_1 and F_0 are matrix polynomials (which do not depend on n) of degrees less than or equal to 2, 1 and 0, respectively. For some of these papers, see [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 18, 19].

In the recent papers [6], [20], orthogonal matrix polynomials of size 2×2 satisfying right-hand side second and fourth order difference equations have been introduced. [6] is in fact devoted to study the algebra of difference operators associated to a family of orthogonal polynomials.

The aim of this paper is to develop a method that leads us to introduce examples of orthogonal matrix polynomials $(P_n)_n$ of arbitrary size satisfying a right-hand side second-order difference equation of the form

$$(3) \quad P_n(x-1)F_{-1}(x) + P_n(x)F_0(x) + P_n(x+1)F_1(x) = \Gamma_n P_n(x), \quad n \geq 0,$$

where F_{-1} , F_0 and F_1 are matrix polynomials which do not depend on n , and Γ_n , $n \geq 0$, are matrices which do not depend on x . In other words, the polynomials P_n , $n \geq 0$, are common eigenfunctions (with left eigenvalues) of the second order difference operator

$$(4) \quad D(\cdot) = \mathfrak{S}_{-1}(\cdot)F_{-1} + \mathfrak{S}_0(\cdot)F_0 + \mathfrak{S}_1(\cdot)F_1,$$

where by \mathfrak{S}_l , $l \in \mathbb{Z}$, we denote the shift operator

$$(5) \quad \mathfrak{S}_l(f) = f(x+l).$$

These families of orthogonal matrix polynomials are among those that are likely to play, in the case of matrix orthogonality, the role of the classical discrete families of Charlier, Meixner, Krawtchouk and Hahn in the case of scalar orthogonality.

The key concept to find our examples is that of symmetric difference operators with respect to a weight matrix W . We say that a difference operator D of the form (4) is symmetric with respect to a weight matrix W if it satisfies $\langle D(P), Q \rangle = \langle P, D(Q) \rangle$ for all matrix polynomials P, Q .

The relationship between symmetric operators with respect to a weight matrix and orthonormal matrix polynomials satisfying second order difference equations is given by the following lemma (Lemma 4.1 of [6]).

Lemma 1.1. *Let W be weight matrix and $(P_n)_n$ a sequence of orthonormal polynomials with respect to W . Let D be a second order difference operator like (4) satisfying that for all polynomial P , $D(P)$ is a polynomial with degree at most the degree of P . Then D is symmetric with respect to W if and only if $D(P_n) = \Lambda_n P_n$ for certain sequence $(\Lambda_n)_n$ of Hermitian matrices.*

In the matrix case, we consider right hand side difference operators with left eigenvalues because of the same reasons that we consider right hand side differential operators with left eigenvalues (2) (see [2] or [7]). It turns out that right-hand side operators as (4) are left linear but not right linear, that is $D(CP) = CD(P)$ with P a matrix function and C a constant matrix, but, in general, $D(PC) \neq PC$. But when we are dealing with an inner product as (1) left linearity is more interesting than right linearity because multiplication on the right by matrices can have terrible consequences on orthogonality. Precisely this is the reason because it is better to consider left eigenvalues: if D satisfies $D(P_n) = P_n \Gamma_n$ the sequence $(D(P_n))_n$ can miss the orthogonality with respect to W . In addition, proceeding as in [2] one can prove that a weight matrix having a second order symmetric right hand side difference operator D with right hand side eigenvalues (and satisfying that for all polynomial P the degree of $D(P)$ is at most the degree of P) actually reduces to a scalar weight (that is the weight matrix is a scalar weight multiplied by a positive definite constant matrix).

The paper is organized as follows. Section 3 expresses the symmetry of a finite difference operator D as a set of difference equations involving the weight matrix W and the coefficients of D plus certain boundary conditions. Section 4 gives a method for solving these equations. The method is based in the assumption that the product $F_1(x-1)F_{-1}(x)$ is a scalar function. Section 5 deals with four special instances satisfying this hypothesis. Using our method we explicitly solve equations in Section 3 to yield interesting families of matrix orthogonal polynomials satisfying second order difference equations of the form (3). These families are classified in accordance with the degree of the matrix coefficient F_1 . For the benefit of the reader, we include here an example for which $\text{dgr}(F_1) = 0$. Let A and J be the $N \times N$ nilpotent and diagonal matrices, respectively, defined by

$$(6) \quad A = \begin{pmatrix} 0 & v_1 & 0 & \cdots & 0 \\ 0 & 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_{N-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad J = \begin{pmatrix} N-1 & 0 & \cdots & 0 \\ 0 & N-2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where v_1, \dots, v_{N-1} are complex numbers.

Theorem 1.2. *Let α be a positive real number. The second order difference operator*

$$(7) \quad D(\cdot) = \alpha \mathfrak{S}_1(\cdot)(I + A) + \mathfrak{S}_0(\cdot) \left(-J - (I + A)^{-1} x \right) + \mathfrak{S}_{-1}(\cdot)(I + A)^{-1} x$$

is symmetric with respect to the weight matrix defined by

$$(8) \quad W = \sum_{x \in \mathbb{N}} \frac{\alpha^x}{x!} (I + A)^x (I + A^*)^x \delta_x.$$

Moreover the monic orthogonal polynomials with respect to W are common eigenfunctions of D with eigenvalues given by

$$\Gamma_n = \alpha(I + A) - J - n(I + A)^{-1}.$$

This paper (together with [6]) shows once again one of the striking developments in this area. While in the scalar case the only possible examples are the well-known families of Charlier, Meixner, Krawtchouk and Hahn, the complexity of the matrix-valued situation, in particular the non commutativity of the matrix product and the existence of singular matrices, opens the door to an embarrassment of riches.

2 Preliminaries. Symmetric difference operators

All the examples considered in Section 5 of this paper are discrete weight matrices of the form

$$(9) \quad W = \sum_{x \in \mathfrak{S}} W(x) \delta_x.$$

For a discrete weight matrix $W = \sum_{x \in \mathfrak{S}} W(x) \delta_x$, supported in a countable set \mathfrak{S} of real numbers, the Hermitian sesquilinear form defined by (1) takes the form $\langle P, Q \rangle = \sum_{x \in \mathfrak{S}} P(x) W(x) Q^*(x)$.

The condition that $\int P(x) dW(x) P^*(x)$ is nonsingular if the leading coefficient of the matrix polynomial P is nonsingular (mentioned at the beginning of this paper) is necessary and sufficient to guarantee the existence of a sequence $(P_n)_n$ of matrix polynomials orthogonal with respect to W , P_n of degree n with nonsingular leading coefficient. For a discrete weight matrix as (9) condition (3) is fulfilled, in particular, when $W(x)$ is positive definite for infinitely many $x \in \mathfrak{S}$. This will be the case of the examples considered in Sections 5.1 and 5.2 below.

If W satisfies that $\int P(x) dW(x) P^*(x)$ is nonsingular if the leading coefficient of the matrix polynomial P is nonsingular and $\text{dgr}(P) \leq K$, for certain nonnegative integer K , one can still associate to W a finite sequence $(P_n)_{n=0}^K$ of orthogonal polynomials. That happens, for instance, if W is supported in finitely many points, and it is the case of the discrete classical families of Krawtchouk and Hahn, or the examples introduced in Sections 5.3 and 5.4 below.

We are interested in weight matrices W which do not reduce to scalar weights. We say that W reduces to scalar weights if there exists a nonsingular matrix T , independent of x , and a diagonal matrix weight $D(x)$ for which

$$W(x) = TD(x)T^*.$$

Diagonal weights (or matrix weights which reduced to scalar weights), as a collection of N scalar weights, belong to the study of scalar orthogonality more

than to the matrix one (this is the case of many examples of orthogonal matrix polynomials which can be found in the literature). For a discrete weight matrix $W = \sum_{x \in \mathfrak{S}} W(x) \delta_x$, supported in a countable set \mathfrak{S} of real numbers, satisfying that $W(a) = I$ for some $a \in \mathfrak{S}$, it is easy to see that W reduces to scalar weights if and only if $W(t)W(s) = W(s)W(t)$ for all $t, s \in \mathfrak{S}$. Using this condition it follows easily that the weight matrices introduced in this paper do not reduce to scalar weights.

The symbol $\mathcal{E}_{i,j}$ stands for the $N \times N$ matrix with entry (i, j) equal to 1 and 0 otherwise. It is easy to see that these matrices satisfy the following property

$$(10) \quad \mathcal{E}_{i,j} \mathcal{E}_{k,l} = \mathcal{E}_{i,l} \delta_{j,k}.$$

We will use that an analytic function f at x_0 , with convergent power series given by

$$f(x) = \sum_{i=0}^{\infty} a_i (x - x_0)^i, \quad |x - x_0| < \epsilon,$$

defines the following function over the matrices M with $\rho(M - x_0 I) < \epsilon$:

$$f(M) = \sum_{i=0}^{\infty} a_i (M - x_0 I)^i.$$

For any two matrices X and Y , we will also use the standard notation

$$\text{ad}_X^0 Y = Y, \quad \text{ad}_X^1 Y = [X, Y] = XY - YX, \quad \text{ad}_X^{n+1} Y = [X, \text{ad}_X^n Y].$$

We will also need the following formula [17, Lemma 5.3, page 160]: if X and Y are $N \times N$ matrices we have

$$(11) \quad e^{-X} Y e^X = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \text{ad}_X^k Y.$$

A finite difference operator can be expanded in terms of several basis. We can use powers of the difference operators Δ and ∇ defined by

$$\Delta(f) = f(x+1) - f(x), \quad \nabla(f) = f(x) - f(x-1).$$

Cross powers $\Delta^i \nabla^j$ are not needed since they are linear combinations of powers of Δ and ∇ (it is an easy consequence of the formula $\Delta \nabla = \Delta - \nabla$). However, we have found more convenient to use the shift operators \mathfrak{S}_l , $l \in \mathbb{Z}$ (see (5)). We can change from the basis Δ^k, ∇^k , $k \geq 0$, to the basis \mathfrak{S}_l , $l \in \mathbb{Z}$, by using the formulas ($k \geq 0$)

$$\begin{aligned} \Delta^k &= \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} \mathfrak{S}_l, & \nabla^k &= \sum_{l=0}^k (-1)^l \binom{k}{l} \mathfrak{S}_{-l}, \\ \mathfrak{S}_k &= \sum_{l=0}^k \binom{k}{l} \Delta^l, & \mathfrak{S}_{-k} &= \sum_{l=0}^k (-1)^l \binom{k}{l} \nabla^l. \end{aligned}$$

Since $\mathfrak{S}_{-1} \circ \mathfrak{S}_1$ is the identity operator, all the shift operators are powers (positive or negative) of \mathfrak{S}_1 .

As we wrote in the Introduction, the key concept in this paper is that of symmetric difference operators. We say that a finite order difference operator D of the form ($s \leq 0 \leq r$)

$$(12) \quad D(\cdot) = \sum_{l=s}^r \mathfrak{S}_l(\cdot) F_l,$$

(where F_l are matrix polynomials) is symmetric with respect to the weight matrix W if $\langle D(P), Q \rangle = \langle P, D(Q) \rangle$ for all matrix polynomials P, Q (where $\langle \cdot, \cdot \rangle$ is defined by (1)).

In [6], one of us has proved that if a finite order difference operator D (12) is symmetric with respect to a weight matrix and the degree of $D(P)$ is at most the degree of P for all matrix polynomials P , then $r = -s$. The symmetry of a finite order difference operator satisfying this condition with respect to a discrete weight matrix W can be guaranteed by a finite set of difference equations together with certain boundary conditions.

Theorem 2.1. *For $r \geq 0$, let D be the finite difference operator*

$$(13) \quad D(\cdot) = \sum_{l=-r}^r \mathfrak{S}_l(\cdot) F_l(x),$$

where $F_l(x)$, $l = -r, \dots, r$ are matrix polynomials. Let W be the discrete weight matrix with support \mathfrak{S} , given by

$$(14) \quad W = \sum_{x \in \mathfrak{S}} W(x) \delta_x.$$

Suppose that the coefficients F_l and the weight matrix W satisfy the following equations

$$(15) \quad F_l(x-l)W(x-l) = W(x)F_{-l}^*(x), \quad \text{for } x \in (l+\mathfrak{S}) \cap \mathfrak{S}, \text{ and } l = 0, 1, \dots, r,$$

and the boundary conditions

$$(16) \quad F_l(x-l)W(x-l) = 0, \quad \text{for } x \in (l+\mathfrak{S}) \setminus \mathfrak{S}, \text{ and } l = 1, \dots, r,$$

$$(17) \quad W(x)F_{-l}^*(x) = 0, \quad \text{for } x \in \mathfrak{S} \setminus (l+\mathfrak{S}), \text{ and } l = 1, \dots, r.$$

Then the difference operator (13) is symmetric with respect to W .

The proof is omitted because it is similar to that found in [5] for scalar difference operators which are symmetric with respect to a measure.

Taking into account Lemma 1.1, if a weight matrix W has a symmetric second order difference operator D such that for any polynomial P , $D(P)$ is a polynomial with degree at most the degree of P , then their orthonormal polynomials $(P_n)_n$ satisfy second order difference equations of the form (3) (and

then any other sequence of orthogonal polynomials with respect to W). The following lemma (Lemma 3.2 of [6]) characterizes the coefficients of those finite difference operators D such that for every polynomial P , $D(P)$ is a polynomial with degree at most the degree of P .

Lemma 2.2. *Let D be a finite difference operator*

$$D(\cdot) = \sum_{l=s}^r \mathfrak{S}_l(\cdot) F_l.$$

The following conditions are equivalent

1. *For any matrix polynomial P , $D(P)$ is also a polynomial with degree at most the degree of P .*
2. *The functions F_l , $l = s, \dots, r$, are polynomials of degree at most $r - s$ and $\text{dgr} \left(\sum_{l=s}^r l^k F_l \right) \leq k$ for $k = 0, \dots, r - s$.*

3 A method for solving the difference equations for the weight matrix

In this paper we are interested in second order difference operators of the form (4). Along the rest of the paper, we consider weight matrices with support $\mathfrak{S} = \{0, 1, 2, \dots, \kappa\}$, with κ either a positive integer, or infinite.

According to Theorem 2.1 for $r = 1$, for the operator $D(\cdot) = \mathfrak{S}_{-1}(\cdot)F_{-1} + \mathfrak{S}_0(\cdot)F_0 + \mathfrak{S}_1(\cdot)F_1$, we have to solve the following equations

$$(18) \quad F_1(x-1)W(x-1) = W(x)F_{-1}^*(x), \quad \text{for } x = 1, 2, \dots, \kappa,$$

$$(19) \quad F_0(x)W(x) = W(x)F_0^*(x), \quad \text{for } x = 0, 1, \dots, \kappa.$$

with the boundary conditions

$$(20) \quad W(0)F_{-1}^*(0) = 0,$$

$$(21) \quad F_1(\kappa)W(\kappa) = 0, \quad \text{if } \kappa \text{ is a positive integer.}$$

We now describe our method for constructing weight matrices W and matrix polynomials F_1 and F_{-1} such that the first order difference equation (18) holds. The method is based on the following assumption on the coefficients F_1 and F_{-1} : there exists a scalar function s such that for $x \in \{1, 2, \dots, \kappa\}$, $s(x) \neq 0$ and

$$F_1(x-1)F_{-1}(x) = |s(x)|^2 I.$$

We now look for a weight matrix W factorized in the form

$$W = \sum_{x=0}^{\kappa} T(x)T^*(x)\delta_x,$$

where T is the matrix function satisfying $T(0) = I$ and the first order difference equation

$$T(x-1) = \frac{F_{-1}(x)}{s(x)}T(x), \quad \text{for } x \in \{1, 2, \dots, \kappa\}.$$

With this choice of W we have

$$\begin{aligned} F_1(x-1)W(x-1) &= F_1(x-1)T(x-1)T^*(x-1) \\ &= F_1(x-1)\frac{F_{-1}(x)}{s(x)}T(x)T^*(x)\frac{F_{-1}^*(x)}{s(x)} \\ &= \frac{F_1(x-1)F_{-1}(x)}{s(x)s(x)}W(x)F_{-1}^*(x) \\ &= W(x)F_{-1}^*(x). \end{aligned}$$

So equation (18) holds.

We have thus proved the following theorem.

Theorem 3.1. *Let κ be either a positive integer or infinite, and F_1 and F_{-1} matrix polynomials. Assume that there exists a scalar function $s(x)$ such that for $x = 1, \dots, \kappa$, $s(x) \neq 0$ and*

$$F_1(x-1)F_{-1}(x) = |s(x)|^2 I, \quad x \in \{1, \dots, \kappa\}.$$

Write T for the solution of the first order difference equation

$$T(x-1) = \frac{F_{-1}(x)}{s(x)}T(x), \quad \text{for } x \in \{1, \dots, \kappa\}, \quad T(0) = I.$$

Then the matrix weight

$$W = \sum_{x=0}^{\kappa} T(x)T^*(x)\delta_x,$$

satisfies the difference equation (18).

4 Four families of illustrative examples

In this section we show four families of orthogonal polynomials satisfying second order difference equations of the form (3) constructed with our method. These families are classified in accordance with the degree of the difference coefficient F_1 .

4.1 Example with $\text{dgr}(F_1) = 0$

Our first example is just the example displayed in Theorem 1.2. The coefficient F_1 of its associated second order difference operator does not depend on x . The example can be considered as a matrix relative of the Charlier scalar weight.

Proof. For this example we have that κ is infinite and

$$\begin{aligned} (22) \quad & F_{-1} = (I + A)^{-1} x, \\ (23) \quad & F_0 = -J - (I + A)^{-1} x, \\ (24) \quad & F_1 = \alpha (I + A). \end{aligned}$$

We now use Theorem 2.1 for $r = 1$ to prove that the operator D is symmetric. We have to check the boundary condition $W(0)F_{-1}^*(0) = 0$ and the equations

$$\begin{aligned} F_1(x-1)W(x-1) &= W(x)F_{-1}^*(x), \quad \text{for } x = 1, 2, \dots \\ F_0(x)W(x) &= W(x)F_0^*(x), \quad \text{for } x = 0, 1, \dots \end{aligned}$$

We proceed in three steps.

First Step. Boundary condition $W(0)F_{-1}^*(0) = 0$.

Since $F_{-1}(x) = (I + A)^{-1} x$, we have $F_{-1}(0) = 0$ and the boundary condition follows straightforwardly.

Second Step. $F_1(x-1)W(x-1) = W(x)F_{-1}^*(x)$ for $x = 1, 2, \dots$.

We use Theorem 3.1 to prove that F_1 , F_{-1} and W satisfy this first order difference equation.

We first check that for $s(x) = \sqrt{\alpha x}$ the matrix polynomials F_{-1} and F_1 satisfy

$$F_1(x-1)F_{-1}(x) = s^2(x)I, \quad x \geq 1.$$

But this is straightforward from the definition of F_{-1} and F_1 (see (22) and (24)).

We then factorize the matrix weight W in the form $W = \sum_{x=0}^{\infty} T(x)T^*(x)\delta_x$, where

$$T(x) = \sqrt{\frac{\alpha^x}{x!}}(I + A)^x.$$

Since $T(0) = I$, we only have to check that for $x \geq 1$, $T(x-1) = \frac{F_{-1}(x)}{s(x)}T(x)$.

Indeed, we have

$$\begin{aligned} \frac{F_{-1}(x)}{s(x)}T(x) &= \frac{(I + A)^{-1} x}{\sqrt{\alpha x}} \sqrt{\frac{\alpha^x}{x!}}(I + A)^x \\ &= \sqrt{\frac{x}{\alpha}} \sqrt{\frac{\alpha^x}{x!}}(I + A)^{-1}(I + A)^x \\ &= \sqrt{\frac{\alpha^{x-1}}{(x-1)!}}(I + A)^{x-1} \\ &= T(x-1). \end{aligned}$$

Theorem 3.1 gives now that F_1 , F_{-1} and W satisfy $F_1(x-1)W(x-1) = W(x)F_{-1}^*(x)$ for $x = 1, 2, \dots$.

Third Step. $F_0(x)W(x) = W(x)F_0^*(x)$ for $x \in \mathbb{N}$.

Since $W(x) = T(x)T^*(x)$ the equation $F_0W = WF_0^*$ is straightforwardly equivalent to the Hermiticity of the function

$$\chi(x) = T^{-1}(x)F_0(x)T(x), \quad x \in \mathbb{N}.$$

We now explicitly compute the function χ to see that actually it is a real diagonal matrix function. To do that, we expand the function $\chi(x)$ in a power series. By writing $S = \log(I + A)$ we get $(I + A)^x = e^{Sx}$. Taking into account that A and S commute, and using formula (11), we have

$$\begin{aligned} \chi(x) &= T^{-1}(x)F_0T(x) = e^{-Sx} (-J - (I + A)^{-1}x) e^{Sx} \\ &= \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{i!} \text{ad}_S^i J x^i - (I + A)^{-1} x. \end{aligned}$$

To compute $\text{ad}_S^i J$, $i \geq 1$, we use the following lemma.

Lemma 4.1. *Let A and J be the matrices given by (6), and let f be an analytic function at 0. Write $S = f(A)$. Then $[S, J] = -Af'(A)$. In particular, for $S = \log(I + A)$ we have $[S, J] = -I + (I + A)^{-1}$.*

Proof. Proof of the lemma.

Write $f(z) = \sum_{k=0}^{\infty} a_k z^k$. It is easy to check that for $k \geq 0$ $[A^k, J] = -kA^k$. Then

$$[S, J] = \sum_{k=0}^{\infty} a_k [A^k, J] = - \sum_{k=0}^{\infty} k a_k A^k = -Af'(A).$$

□

Using this lemma we get

$$\text{ad}_S J = [S, J] = (I + A)^{-1} - I.$$

For $\text{ad}_S^i J$, $i \geq 2$, since S commutes with A , we have

$$\text{ad}_S^i J = \text{ad}_S^{i-1} (\text{ad}_S J) = \text{ad}_S^{i-1} ((I + A)^{-1} - I) = 0.$$

Hence, we get

$$\chi(x) = -J - xI.$$

That is, the matrix function χ is real diagonal, and then Hermitian.

Thus, Theorem 2.1 for $r = 1$ shows that the second order operator D (7) is symmetric with respect to W .

We now use Lemma 1.1 to prove that the orthogonal polynomials are common eigenfunctions of the operator D . Since we have already proved that D is symmetric with respect to W , it is enough to check that for any polynomial P

the degree of $D(P)$ is at most the degree of P . But that is a consequence of Lemma 2.2 taking into account that

$$\begin{aligned} \text{dgr}(F_i) &\leq 2, \quad \text{for } i = 0, 1, -1, \\ \text{dgr}(F_1 - F_{-1}) &= \text{dgr}(\alpha(I + A) - (I + A)^{-1}x) = 1, \\ \text{dgr}(F_0 + F_1 + F_{-1}) &= \text{dgr}(-J + \alpha(I + A)) = 0. \end{aligned}$$

The expression for the eigenvalues follows by applying Theorem 3.3 of [6].

The proof of the theorem is now complete. \square

4.2 Example with $\text{dgr}(F_1) = 1$ and unbounded support.

Our second example can be considered as a matrix relative of the Meixner scalar weight. It has unbounded support and the coefficient F_1 of its associated second order difference operator is a polynomials of degree 1.

Theorem 4.2. *Let A and J be the $N \times N$ nilpotent and diagonal matrices, respectively, given by (6). Given two positive real numbers μ and γ , with $0 < \mu < 1$, we consider the matrix*

$$(25) \quad R_{A,\mu} = (I - A)(I - \mu A)^{-1},$$

and the weight matrix defined by

$$(26) \quad W = \frac{1}{\Gamma(\gamma)} \sum_{x=0}^{\infty} \frac{\mu^x \Gamma(x + \gamma)}{x!} R_{A,\mu}^x (R_{A,\mu}^*)^x \delta_x.$$

Then the second order difference operator

$$(27) \quad D(\cdot) = \mathfrak{S}_1(\cdot)F_1(x) + \mathfrak{S}_0(\cdot)F_0(x) + \mathfrak{S}_{-1}(\cdot)F_{-1}(x),$$

where

$$(28) \quad F_{-1}(x) = R_{A,\mu}^{-1}x,$$

$$(29) \quad F_0(x) = (\mu - 1)J - x(\mu R_{A,\mu} + R_{A,\mu}^{-1}),$$

$$(30) \quad F_1(x) = \mu(x + \gamma)R_{A,\mu},$$

is symmetric with respect to the weight matrix W . Moreover, the monic orthogonal polynomials with respect to W are common eigenfunctions of D , with eigenvalues given by

$$\Gamma_n = (\mu - 1)J + \mu\gamma R_{A,\mu} + n(\mu R_{A,\mu} - R_{A,\mu}^{-1}).$$

Proof. We use again Theorem 2.1 for $r = 1$ to prove that the operator D is symmetric. We have to check the boundary condition $W(0)F_{-1}^*(0) = 0$ and the equations

$$(31) \quad F_1(x-1)W(x-1) = W(x)F_{-1}(x), \quad \text{for } x = 1, 2, \dots$$

$$(32) \quad F_0(x)W(x) = W(x)F_0^*(x), \quad \text{for } x = 0, 1, \dots$$

We omit the proofs of the boundary condition and the equation (31) because they are similar to those in Steps 1 and 2 in Theorem 1.2 (just taking here $s(x) = \sqrt{\mu(x + \gamma - 1)x}$). To prove $F_0(x)W(x) = W(x)F_0^*(x)$ for $x \in \mathbb{N}$, we proceed as follows. We first write $W(x) = T(x)T^*(x)$, where

$$T(x) = \sqrt{\frac{\mu^x \Gamma(x + \gamma)}{\Gamma(\gamma)x!}} R_{A,\mu}^x.$$

As we have already pointed out, the equation $F_0(x)W(x) = W(x)F_0^*(x)$ is then equivalent to the Hermiticity of the matrix function $\chi(x) = T^{-1}(x)F_0(x)T(x)$. We are going to show that this is, in fact, a real diagonal matrix.

Consider now the analytic function at 0

$$f(z) = \log\left(\frac{1-z}{1-\mu z}\right),$$

and write $S = f(A)$. This gives $e^S = R_{A,\mu}$ and $e^{-S} = R_{A,\mu}^{-1}$, and hence

$$\begin{aligned} \chi(x) &= T^{-1}(x)F_0(x)T(x) \\ &= e^{-Sx}((\mu - 1)J - \mu e^S x - e^{-S}x) e^{Sx} \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} (\mu - 1) \operatorname{ad}_S^i J x^i - \mu e^S x - e^{-S}x. \end{aligned}$$

We now compute explicitly the function χ

Using Lemma 4.1, we see that $-(\mu - 1)[S, J] - \mu e^S - e^{-S}$ is, in fact, a multiple of the identity matrix. Indeed, since

$$f'(z) = \frac{\mu}{1-\mu z} - \frac{1}{1-z},$$

according to this lemma, we have

$$[S, J] = A(1 - A)^{-1} - \mu A(1 - \mu A)^{-1}.$$

On the other hand, a simple computation gives

$$-\mu e^S - e^{-S} = -(\mu + 1)I - \mu(\mu - 1)A(1 - \mu A)^{-1} + (\mu - 1)A(I - A)^{-1}.$$

Then we get

$$-(\mu - 1)[S, J] - \mu e^S - e^{-S} = -(1 + \mu)I.$$

For the rest of the coefficients, we proceed as in Theorem 1.2 to get $\operatorname{ad}_R^i J = 0$ for $i \geq 2$.

We have then that

$$\chi(x) = (\mu - 1)J - (1 + \mu)Ix.$$

This is a real diagonal matrix for all $x \geq 0$, and so Hermitian. Then $F_0(x)W(x) = W(x)F_0^*(x)$ holds for $x \in \mathbb{N}$.

Using Theorem 2.1 for $r = 1$ we can conclude that the second order difference operator (27) is symmetric with respect to the weight matrix W (26).

We now use Lemma 1.1 to see that the matrix orthogonal polynomials with respect to W are common eigenfunctions of the difference operator (27). Since we have already proved the symmetry of D , we have just to see that for any polynomial P , $D(P)$ has degree at most the degree of P . This is a consequence of Lemma 2.2 since

$$\begin{aligned} \text{dgr}(F_i) &\leq 2, \quad \text{for } i = 0, 1, -1, \\ \text{dgr}(F_1 - F_{-1}) &= \text{dgr}\left(\mu(x + \gamma)R_{A,\mu} - R_{A,\mu}^{-1}x\right) = 1, \\ \text{dgr}(F_0 + F_1 + F_{-1}) &= \text{dgr}\left((\mu - 1)J + \mu\gamma R_{A,\mu}\right) = 0. \end{aligned}$$

The expression for the eigenvalues follows by [6], Theorem 3.3.

The proof of the theorem is now complete. □

4.3 Example with $\text{dgr}(F_1) = 1$ and finite support.

Our third example can be considered as a matrix relative of the Krawtchouk scalar weight. It has then finite support and the coefficient F_1 of its associated second order difference operator is a polynomial of degree 1 (the proof will be omitted because is similar to that of theorems 1.2 and 4.2)

Theorem 4.3. *Let A and J be the $N \times N$ matrices given by (6). Given a positive integer κ and a positive real number μ , we consider the matrix*

$$(33) \quad R_{A,\mu} = (I + A)(I - \mu A)^{-1},$$

and the matrix weight defined by

$$(34) \quad W = \sum_{x=0}^{\kappa} \frac{\mu^x \Gamma(\kappa + 1)}{\Gamma(\kappa + 1 - x)x!} R_{A,\mu}^x (R_{A,\mu}^*)^x \delta_x.$$

Then the second order difference operator

$$(35) \quad D(\cdot) = \mathfrak{S}_1(\cdot)F_1(x) + \mathfrak{S}_0(\cdot)F_0(x) + \mathfrak{S}_{-1}(\cdot)F_{-1}(x),$$

where

$$\begin{aligned} F_{-1}(x) &= R_{A,\mu}^{-1}x, \\ F_0(x) &= -(\mu + 1)J + \mu R_{A,\mu}x - R_{A,\mu}^{-1}x, \\ F_1(x) &= -\mu R_{A,\mu}x + \mu\kappa R_{A,\mu}, \end{aligned}$$

is symmetric with respect to W . Moreover, the monic orthogonal polynomials with respect to W are common eigenfunctions of D and its eigenvalues are given by

$$\Gamma_n = -(\mu + 1)J + \mu\kappa R_{A,\mu} - n\left(\mu R_{A,\mu} + R_{A,\mu}^{-1}\right).$$

4.4 Example with $\text{dgr}(F_1) = 2$

Our third example seems not to have any scalar relative. The weight matrix has bounded support and the difference coefficient F_1 has degree two.

For a positive integer κ and complex numbers $v_i \in \mathbb{C}$, $1 \leq i \leq N-1$, we define the nilpotent matrices (with order of nilpotency 2 and $N-1$ respectively)

$$(36) \quad B = \frac{-2v_1}{\kappa+1} \mathcal{E}_{1,2},$$

$$(37) \quad C = v_2 \mathcal{E}_{1,3} + \sum_{\substack{i=1 \\ i \neq 2}}^{N-1} v_i \mathcal{E}_{i,i+1}.$$

We also define the diagonal matrix

$$(38) \quad L = \mathcal{E}_{1,1} + 5\mathcal{E}_{2,2} + \sum_{i=3}^N (2i-3) \mathcal{E}_{i,i}.$$

It is a matter of computation to check that these matrices satisfy the following equations

$$(39) \quad B^2 = 0,$$

$$(40) \quad BC = CB = 0,$$

$$(41) \quad \text{ad}_C L = 2C - (\kappa+1)B,$$

$$(42) \quad \text{ad}_{C^k} L = 2kC^k, \quad \text{for } k \geq 2,$$

$$(43) \quad \text{ad}_B L = 4B.$$

We are now ready to introduce our last example.

Theorem 4.4. *Let B , C and L be the matrices defined by (36), (37) and (38) respectively. We consider the matrix*

$$(44) \quad R_C = \left(I - \frac{1}{2}C \right) \left(I + \frac{1}{2}C \right)^{-1}$$

and the weight matrix defined by

$$(45) \quad W = \sum_{x=0}^{\kappa} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-x)x!} \left(R_C^x - B \frac{x(x+1)}{2} \right) \left((R_C^*)^x - B^* \frac{x(x+1)}{2} \right) \delta_x.$$

Then the second order difference operator

$$(46) \quad D(\cdot) = \mathfrak{S}_1(\cdot)F_1(x) + \mathfrak{S}_0(\cdot)F_0(x) + \mathfrak{S}_{-1}(\cdot)F_{-1}(x)$$

where

$$(47) \quad F_1 = Bx^2 + ((1-\kappa)B - R_C)x + \kappa(R_C - B),$$

$$(48) \quad F_0 = -2Bx^2 - (R_C^{-1} + (1-\kappa)B - R_C)x + L,$$

$$(49) \quad F_{-1} = Bx^2 + R_C^{-1}x,$$

is symmetric with respect to the weight matrix W (45). Moreover, the monic orthogonal polynomials with respect to W are common eigenfunctions of D and its eigenvalues are given by

$$\Gamma_n = L + \kappa(R_C - B) + n((1 - \kappa)B - R_C - R_C^{-1}) + n(n - 1)B.$$

Proof. We use Theorem 2.1 for $r = 1$ to prove the symmetry of D with respect to W . For this we have to check the boundary conditions, $F_1(\kappa)W(\kappa) = 0$ and $W(0)F_{-1}^*(0) = 0$ and the equations

$$\begin{aligned} F_1(x - 1)W(x - 1) &= W(x)F_{-1}^*(x), \quad \text{for } x = 1, \dots, \kappa \\ F_0(x)W(x) &= W(x)F_0^*(x), \quad \text{for } x = 0, 1, \dots, \kappa. \end{aligned}$$

We skip over the proof of the boundary conditions and the first symmetry equation since it is similar to that for Theorem 1.2 (taking now $s(x) = \sqrt{x(\kappa + 1 - x)}$).

We write $W = \sum_{x=0}^{\kappa} T(x)T^*(x)\delta_x$ for

$$T(x) = \sqrt{\frac{\Gamma(\kappa + 1)}{\Gamma(\kappa + 1 - x)x!}} \left(R_C^x - B \frac{x(x + 1)}{2} \right).$$

Then the equation $F_0(x)W(x) = W(x)F_0^*(x)$ is equivalent to the Hermiticity of the matrix function

$$\chi(x) = T^{-1}(x)F_0(x)T(x).$$

We again compute explicitly the function χ to see that it is in fact a real diagonal matrix. Consider the analytic function at 0

$$(50) \quad f(z) = \log \left(\frac{1 + \frac{z}{2}}{1 - \frac{z}{2}} \right),$$

and write $S = f(C)$. This gives $R_C^{-1} = e^S$ and $R_C = e^{-S}$. The function $f(C)$ only contains odd powers of C , then by property (40) we get that $SB = BS = 0$, and so

$$e^{Sx}B = Be^{Sx} = B = e^{-Sx}B = Be^{-Sx}.$$

Taking into account this remark as well as the property (39) we can write

$$\begin{aligned} T^{-1}(x) &= \sqrt{\frac{\Gamma(\kappa + 1 - x)x!}{\Gamma(\kappa + 1)}} \left(e^{-Sx} - B \frac{x(x + 1)}{2} \right)^{-1} \\ &= \sqrt{\frac{\Gamma(\kappa + 1 - x)x!}{\Gamma(\kappa + 1)}} \left(\left(I - B \frac{x(x + 1)}{2} \right) e^{-Sx} \right)^{-1} \\ &= \sqrt{\frac{\Gamma(\kappa + 1 - x)x!}{\Gamma(\kappa + 1)}} \left(I + B \frac{x(x + 1)}{2} \right) e^{Sx}. \end{aligned}$$

Because of (39) and (43), we have $\text{ad}_B^i L = 0$ for $i \geq 2$. Taking into account all these properties we can write $\chi(x)$ as the power series

$$(51) \quad \begin{aligned} \chi(x) = & L + \left((\kappa - 1)B + e^{-S} - e^S + \text{ad}_S L + \frac{1}{2} \text{ad}_B L \right) x \\ & + \left(-2B + \frac{1}{2} \text{ad}_B (L + \text{ad}_S L) + \frac{1}{2} \text{ad}_S^2 L \right) x^2 \\ & + \sum_{i=3}^{\infty} \left(\frac{1}{2} \text{ad}_B \left(\frac{\text{ad}_S^{i-1} L}{(i-1)!} + \frac{\text{ad}_S^{i-2} L}{(i-2)!} \right) + \frac{\text{ad}_S^i L}{i!} \right) x^i. \end{aligned}$$

We now prove that, except the first one, all the coefficients above vanish. To do that, we use the following lemma.

Lemma 4.5. *Let B , C and L be the matrices defined by (36), (37) and (38). For an analytic function f at 0 the following equations hold*

$$(52) \quad ((\kappa + 1)f'(0) - 2)B + \text{ad}_{f(C)} L + \frac{1}{2} \text{ad}_B L = 2Cf'(C),$$

$$(53) \quad \text{ad}_{f(C)}^{i+1} L = 0, \quad \text{for } i \geq 1,$$

$$(54) \quad \text{ad}_B (\text{ad}_{f(C)} L) = 0.$$

Proof. Write $f(z) = \sum_{k \geq 0} a_k z^k$. Taking into account equations (41) and (42), we get

$$\text{ad}_{f(C)} L = \sum_{k \geq 0} a_k [C^k, L] = 2 \sum_{k \geq 0} a_k k C^k - (\kappa + 1)a_1 B = 2Cf'(C) - (\kappa + 1)f'(0)B,$$

and then using equation (43) we get

$$\begin{aligned} ((\kappa + 1)f'(0) - 2)B + \text{ad}_{f(C)} L + \frac{1}{2} \text{ad}_B L = & ((\kappa + 1)f'(0) - 2)B + 2Cf'(C) \\ & - (\kappa + 1)f'(0)B + 2B = 2Cf'(C). \end{aligned}$$

Since $BC = CB = 0$, any power of C commute with B , then we have

$$\begin{aligned} \text{ad}_{f(C)}^{i+1} L = \text{ad}_{f(C)}^i (2Cf'(C) - (\kappa + 1)a_1 B) = & 0, \quad \text{for } i \geq 1, \\ \text{ad}_B (\text{ad}_{f(C)} L) = \text{ad}_B (2Cf'(C) - (\kappa + 1)a_1 B) = & 0. \end{aligned}$$

□

Using now equation (52) of Lemma 4.5 for the function f in (50) (for which $f'(0) = 1$) and $S = f(C)$, we get

$$(55) \quad \begin{aligned} (\kappa - 1)B + e^{-S} - e^S + \text{ad}_S L + \frac{1}{2} \text{ad}_B L = & e^{-S} - e^S + 2Cf'(C) \\ = e^{-S} - e^S + \left(\left(I + \frac{C}{2} \right)^{-1} + \left(I - \frac{C}{2} \right)^{-1} \right) & \\ = 0. & \end{aligned}$$

On the other hand, using equations (53) and (54) of Lemma 4.5, we obtain

$$(56) \quad -2B + \frac{1}{2} \operatorname{ad}_B (L + \operatorname{ad}_S L) + \frac{1}{2} \operatorname{ad}_S^2 L = 0,$$

$$(57) \quad \frac{1}{2} \operatorname{ad}_B \left(\frac{\operatorname{ad}_S^{i-1} L}{(i-1)!} + \frac{\operatorname{ad}_S^{i-2} L}{(i-2)!} \right) + \frac{\operatorname{ad}_S^i L}{i!} = 0, \quad i \geq 3,$$

where we have also used the equation (43).

Equations (51), (55), (56) and (57) then give that $\chi(x) = L$, which it is a real diagonal matrix. So $F_0(x)W(x) = W(x)F_0(x)$ for $x = 0, 1, \dots, \kappa$.

By Theorem 2.1 for $r = 1$, we conclude that D is symmetric with respect to W .

To see that the orthogonal polynomials with respect to W are common eigenfunctions of D , we use Theorem 1.1. Since we have already proved that D is symmetric with respect to W , it remains to see that for any polynomial P , $D(P)$ is a polynomial with degree at most the degree of P . This is a consequence of Lemma 2.2, since

$$\begin{aligned} \operatorname{dgr}(F_i) &\leq 2 \quad \text{for } i = -1, 0, 1, \\ \operatorname{dgr}(F_1 - F_{-1}) &= \operatorname{dgr}(((1 - \kappa)B - e^{-S} - e^S)x + \kappa(e^{-S} - B)) = 1, \\ \operatorname{dgr}(F_0 + F_1 + F_{-1}) &= \operatorname{dgr}(L + \kappa(e^{-S} - B)) = 0. \end{aligned}$$

To obtain the expression for the eigenvalues we apply [6], Theorem 3.3.

The proof of the theorem is now complete. \square

References

- [1] M. J. Cantero, L. Moral and L. Velázquez, Matrix orthogonal polynomials whose derivatives are also orthogonal, *J. Approx. Theory*, **146** (2007), 174–211.
- [2] A. J. Durán, Matrix inner product having a matrix symmetric second order differential operator, *Rocky Mount. J. Math.* **27** (1997), 585–600.
- [3] A. J. Durán, A method to find weight matrices having symmetric second order differential operators with matrix leading coefficient, *Constr. Approx.* **29** (2009), 181–205.
- [4] A. J. Durán, Rodrigues' Formulas for orthogonal matrix polynomials satisfying higher order differential equations, *Experiment. Math.* **20** (2011), 15–24.
- [5] A. J. Durán, Orthogonal polynomials satisfying higher order difference equations, to appear in *Constr. Approx.*
- [6] A. J. Durán, The algebra of difference operators associated to a family of orthogonal polynomials, *J. Approx. Th.* **164** (2012) 586–610.

- [7] A. J. Durán and F. A. Grünbaum, Orthogonal matrix polynomials satisfying second order differential equations, *Int. Math. Res. Not.* **2004:10** (2004), 461–484.
- [8] A. J. Durán and F. A. Grünbaum, Structural formulas for orthogonal matrix polynomials satisfying second order differential equations, I, *Constr. Approx.* **22** (2005), 255–271.
- [9] A. J. Durán and F. A. Grünbaum, Matrix orthogonal polynomials satisfying second order differential equations: coping without help from group representation theory, *J. Approx. Theory* **148** (2007), 35–48.
- [10] A. J. Durán and M. D. de la Iglesia, Some examples of orthogonal matrix polynomials satisfying odd order differential equations, *J. Approx. Theory* (2008), **150**, 153–174.
- [11] A. J. Durán and M. D. de la Iglesia, Second-order differential operators having several families of orthogonal matrix polynomials as eigenfunctions, *Int. Math. Res. Not.* **2008** doi:10.1093/imrn/rnn084.
- [12] F. A. Grünbaum, Matrix valued Jacobi polynomials, *Bull. Sciences Math.* **127**, (2003) 207–214.
- [13] F. A. Grünbaum, I. Pacharoni and J. Tirao, Matrix valued spherical functions associated to the complex projective plane, *J. Funct. Anal.* **188** (2002), 350–441.
- [14] F. A. Grünbaum, I. Pacharoni and J. Tirao, Matrix valued orthogonal polynomials of the Jacobi type, *Indag. Mathem.* **14** (2003), 353–366.
- [15] M.G. Krein, Fundamental aspects of the representation theory of Hermitian operators with deficiency index (m, m) , *Ukrain. Math. Zh.* **1** (1949), 3–66; *Amer. Math. Soc. Transl. (2)* **97** (1970), 75–143.
- [16] M.G. Krein, Infinite J-matrices and a matrix moment problem, *Dokl. Akad. Nauk SSSR* **69** (1949) , 125–128.
- [17] W. Miller, *Symmetry Groups and their Applications*, Academic Press, New York, 1972
- [18] I. Pacharoni and P. Román, A sequence of matrix valued orthogonal polynomials associated to spherical functions, *Constr. Approx.*, **28**, (2008) 127–147.
- [19] I. Pacharoni and J. A. Tirao, Matrix valued orthogonal polynomials arising from the complex projective space, *Constr. Approx.* **25**, (2006) 177–192.
- [20] L. Vinet and A. Zhedanov, Representations of the Schrödinger group and matrix orthogonal polynomials, *J. Phys. A: Math. Theor.* **44**, (2011) 355201 (28pp).