Journal of Russian Laser Research, Volume 27, Number 1, 2006

A q-ANALOG OF RACAH POLYNOMIALS AND q-ALGEBRA $SU_q(2)$ IN QUANTUM OPTICS

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Abstract

We study some q-analogs of Racah polynomials and some of their applications in the theory of representation of quantum algebras. Possible implementations in quantum optics are discussed.

Keywords: representation theory, deformations, quantum groups, q-algebra.

1. Introduction

The symmetries of quantum systems devoted to Lie groups play an important role in explaining the properties of different states of atoms and photons [1, 2]. During the last decade, quantum groups and their representations were considered to be employed in describing the properties of quantum states of atoms and photons [3]. Quantum groups are generalizations of usual Lie groups like the rotation group or $SU(2)$-group. The standard properties of the rotation-group representation and their characteristics like Racah coefficients and $6j$-symbols are well known [4]. One needs to develop also the mathematical formalism of quantum groups to apply them in quantum optics and quantum mechanics.

An orthogonal polynomial family that generalizes the Racah coefficients or $6j$-symbols (so-called Racah and q-Racah polynomials) was introduced in [5]. These polynomials are at the top of the so-called Askey scheme (see, e.g., [6]) that contains all classical families of hypergeometric orthogonal polynomials. Some years later the same authors [7] introduced the celebrated Askey–Wilson polynomials. The important property of these polynomials is the possibility to obtain from them all known families of hypergeometric polynomials and q-polynomials as particular or limit cases (the review is done in the nice survey [6]). The main tool of [6, 7] was the hypergeometric and basic series, respectively. On the other hand, in [8] (see also [9]) q-polynomials were considered as the solution of a second-order difference equation of the hypergeometric type on the nonlinear lattice,

$$x(s) = c_1 q^s + c_2 q^{-s} + c_3.$$

In particular, it was shown that the solution of the hypergeometric-type equation can be expressed as certain basic series and, in such a way, the results by Askey and Wilson were recovered.

Manuscript submitted by the authors in English November 25, 2005.

The interest in such polynomials increases after the appearance of q-algebras and quantum groups [10–14]. However, from the first attempts to construct the q-analog of the Wigner–Racah formalism for the simplest quantum algebra $U_q(su(2))$ [15] (see also [16–18]), it becomes clear that for obtaining the q-polynomials intimately connected with the q-analogs of the Racah and Clebsh–Gordan coefficients, i.e., a q-analog of the Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ and the dual Hahn polynomials $w_n^c(x(s),a,b)_q$, respectively, it is better to use a different lattice. In fact, the q-Racah polynomials $R_n^{\beta,\gamma}(x(s),N,\delta)_q$ introduced in [7] (see also [6]) were defined on the lattice

$$x(s) = q^{-s} + \delta q^{-N} q^s$$

that depends not only on the variable s but also on the parameters of the polynomials, namely,

$$x(s) = [s]_q[s+1]_q \tag{1}$$

that, in turn, depends only on s, where by $[s]_q$ we denote the q-numbers (in its symmetric form)

$$[s]_q = \frac{q^{s/2} - q^{-s/2}}{q^{1/2} - q^{-1/2}}, \quad \forall s \in \mathbb{C}. \tag{2}$$

With this choice, the q-Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ are proportional to the q-Racah coefficients (or $6j$-symbols) of the quantum algebra $U_q(su(2))$. A very nice and simple approach to $6j$-symbols has been developed recently in [19].

Moreover, this connection gives the possibility to a deeper study of the Wigner–Racah formalism (or the q-analog of the quantum theory of angular momentum [20–23]) for the quantum algebras $U_q(su(2))$ and $U_q(su(1,1))$ using the powerful and well-known theory of orthogonal polynomials on nonuniform lattices. On the other hand, using the q-analog of the quantum theory of angular momentum [20–23] we can obtain several results for the q-polynomials, some of which are nontrivial from the viewpoint of the theory of orthogonal polynomials (see, e.g., the nice surveys [24, 25]). In fact, in this paper we present a detailed study of some q-analogs of the Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ and $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ on the lattice (1), as well as their connection with the q-Racah coefficients (or $6j$-symbols) of the quantum algebra $U_q(su(2))$, in order to establish which properties of the polynomials correspond to the $6j$-symbols and vice versa.

The structure of the paper is as follows.

In Sec. 2, we present some general results of the theory of orthogonal polynomials on nonuniform lattices adopted from [9, 26]. In Sec. 2.1, a detailed discussion of the Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ is done, whereas in Sec. 2.2, the $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ are considered; in particular, a relation between these families is established. In Sec. 3, a comparative analysis of such families and $6j$-symbols of the quantum algebra $U_q(su(2))$ is developed which gives, on one hand, some information on the Racah coefficients and, on the other hand, allows one to give a group-theoretical interpretation of the Racah polynomials on the lattice (1). Finally, some comments and remarks on q-Racah polynomials and the quantum algebra $U_q(su(3))$ as well as possible applications in the models of photon–atom interactions are included.

2. Some General Properties of q-Polynomials

We will start with some general properties of orthogonal hypergeometric polynomials on nonuniform lattices [9, 27].

The hypergeometric polynomials are the polynomial solutions $P_n(x(s))_q$ of the second-order linear difference equation of the hypergeometric type on a nonuniform lattice $x(s)$ (SODE)

$$\sigma(s)\frac{\Delta}{\Delta x(s-\frac{1}{2})}\frac{\nabla y(s)}{\nabla x(s)}+\tau(s)\frac{\Delta y(s)}{\Delta x(s)}+\lambda y(s)=0, \quad x(s)=c_1[q^s+q^{-s-\mu}]+c_3, \quad q^\mu=\frac{c_1}{c_2},$$
$$\nabla f(s)=f(s)-f(s-1), \qquad \Delta f(s)=f(s+1)-f(s), \tag{3}$$

or, equivalently,

$$A_s y(s+1)+B_s y(s)+C_s y(s-1)+\lambda y(s)=0, \tag{4}$$

where

$$A_s=\frac{\sigma(s)+\tau(s)\Delta x(s-\frac{1}{2})}{\Delta x(s)\Delta x(s-\frac{1}{2})}, \quad C_s=\frac{\sigma(s)}{\nabla x(s)\Delta x(s-\frac{1}{2})}, \quad B_s=-(A_s+C_s).$$

Notice that $x(s)=x(-s-\mu)$.

In the following, we will use the notation[1]

$$P_n(s)_q:=P_n(x(s))_q, \quad \sigma(-s-\mu)=\sigma(s)+\tau(s)\Delta x(s-\tfrac{1}{2}).$$

With this notation, Eq. (3) becomes

$$\sigma(-s-\mu)\frac{\Delta P_n(s)_q}{\Delta x(s)}-\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)}+\lambda_n\Delta x(s-\tfrac{1}{2})P_n(s)_q=0\,. \tag{5}$$

The polynomial solutions $P_n(s)_q$ of (3) can be obtained by the following Rodrigues-type formula [9, 28]:

$$P_n(s)_q=\frac{B_n}{\rho(s)}\nabla^{(n)}\rho_n(s), \qquad \nabla^{(n)}:\frac{\nabla}{\nabla x_1(s)}\frac{\nabla}{\nabla x_2(s)}\cdots\frac{\nabla}{\nabla x_n(s)}, \tag{6}$$

where

$$x_m(s)=x\left(s+\frac{m}{2}\right)$$
$$\rho_n(s)=\rho(s+n)\prod_{m=1}^{n}\sigma(s+m), \tag{7}$$

and $\rho(s)$ is a solution of the Pearson-type equation

$$\Delta\left[\sigma(s)\rho(s)\right]=\tau(s)\rho(s)\Delta x(s-1/2),$$

or equivalently,

$$\frac{\rho(s+1)}{\rho(s)}=\frac{\sigma(s)+\tau(s)\Delta x(s-\frac{1}{2})}{\sigma(s+1)}=\frac{\sigma(-s-\mu)}{\sigma(s+1)}. \tag{8}$$

Let us point out that the function ρ_n satisfies the equation

$$\Delta\left[\sigma(s)\rho_n(s)\right]=\tau_n(s)\rho_n(s)\Delta x_n\left(s-\frac{1}{2}\right),$$

[1] In the exponential lattice $x(s)=c_1q^{\pm s}+c_3$, so $\mu=\pm\infty$; therefore, instead of using $\sigma(-s-\mu)$ one should use the equivalent function $\sigma(s)+\tau(s)\Delta x(s-\frac{1}{2})$.

where $\tau_n(s)$ is given by

$$\tau_n(s)=\frac{\sigma(s+n)+\tau(s+n)\Delta x(s+n-\frac{1}{2})-\sigma(s)}{\Delta x_{n-1}(s)}=\frac{\sigma(-s-n-\mu)-\sigma(s)}{\Delta x_n(s-\frac{1}{2})}=\tau_n' x_n(s)+\tau_n(0) \qquad (9)$$

and

$$\tau_n'=-\frac{\lambda_{2n+1}}{[2n+1]_q}, \qquad \tau_n(0)=\frac{\sigma(-s_n^\star-n-\mu)-\sigma(s_n^\star)}{x_n(s_n^\star+\frac{1}{2})-x_n(s_n^\star-\frac{1}{2})},$$

with $s_n^\star$ being the zero of the function $x_n(s)$, i.e., $x_n(s_n^\star)=0$.

From (6) an explicit formula for the polynomials P_n follows (see Eq. (3.2.30) in [9]):

$$P_n(s)_q=B_n\sum_{m=0}^{n}\frac{[n]_q!(-1)^{m+n}}{[m]_q![n-m]_q!}\frac{\nabla x(s+m-\frac{n-1}{2})}{\prod_{l=0}^{n}\nabla x(s+\frac{m-l+1}{2})}\frac{\rho_n(s-n+m)}{\rho(s)}, \qquad (10)$$

where $[n]_q$ denotes the symmetric q-numbers (2) and the q-factorials are given by

$$[0]_q!:=1, \quad [n]_q!:=[1]_q[2]_q\cdots[n]_q, \quad n\in\mathbb{N}.$$

It can be shown [9, 27, 28] that the most general polynomial solution of the q-hypergeometric equation (3) corresponds to

$$\sigma(s)=A\prod_{i=1}^{4}[s-s_i]_q=Cq^{-2s}\prod_{i=1}^{4}(q^s-q^{s_i}), \qquad A\cdot C\neq 0 \qquad (11)$$

and has the form (see Eq. (49a) in [28], p. 240)

$$P_n(s)_q=D_n\ {}_4\phi_3\left(\begin{array}{c}q^{-n},q^{2\mu+n-1+\sum_{i=1}^4 s_i},q^{s_1-s},q^{s_1+s+\mu}\\ q^{s_1+s_2+\mu},q^{s_1+s_3+\mu},q^{s_1+s_4+\mu}\end{array};q\,,\,q\right), \qquad (12)$$

where the normalizing factor D_n is given by ($\varkappa_q:=q^{1/2}-q^{-1/2}$)

$$D_n=B_n\left(\frac{-A}{c_1q^\mu\varkappa_q^5}\right)^n q^{-\frac{n}{2}(3s_1+s_2+s_3+s_4+\frac{3(n-1)}{2})}(q^{s_1+s_2+\mu};q)_n(q^{s_1+s_3+\mu};q)_n(q^{s_1+s_4+\mu};q)_n.$$

The basic hypergeometric series ${}_r\phi_p$ are defined by [6]

$${}_r\phi_p\left(\begin{array}{c}a_1,\dots,a_r\\ b_1,\dots,b_p\end{array};q\,,\,z\right)=\sum_{k=0}^{\infty}\frac{(a_1;q)_k\cdots(a_r;q)_k}{(b_1;q)_k\cdots(b_p;q)_k}\frac{z^k}{(q;q)_k}\left[(-1)^kq^{\frac{k}{2}(k-1)}\right]^{p-r+1},$$

where $(a;q)_k=\prod_{m=0}^{k-1}(1-aq^m)$ is the q-analog of the Pocchammer symbol.

In this paper, we deal with orthogonal q-polynomials and functions.

It can be proven [9], in view of the difference equation of the hypergeometric type (3), that, if the boundary conditions

$$\sigma(s)\rho(s)x^k(s-1/2)\big|_{s=a,b}=0\forall k\geq 0$$

hold, then the polynomials $P_n(s)_q$ are orthogonal with respect to the weight function ρ, i.e.,

$$\sum_{s=a}^{b-1} P_n(s)_q P_m(s)_q \rho(s)\Delta x(s-1/2) = \delta_{nm} d_n^2, \quad s = a, a+1, \dots, b-1. \tag{13}$$

The squared norm in (13) reads (see Eq. (3.7.15) in [9])

$$d_n^2 = (-1)^n A_{n,n} B_n^2 \sum_{s=a}^{b-n-1} \rho_n(s)\Delta x_n(s-1/2), \tag{14}$$

where (see [9], p. 66)

$$A_{n,k} = \frac{[n]_q!}{[n-k]_q!} \prod_{m=0}^{k-1} \left(-\frac{\lambda_{n+m}}{[n+m]_q} \right). \tag{15}$$

A simple consequence of the orthogonality is the three-term recurrence relation (TTRR)

$$x(s) P_n(s)_q = \alpha_n P_{n+1}(s)_q + \beta_n P_n(s)_q + \gamma_n P_{n-1}(s)_q, \tag{16}$$

where α_n, β_n, and γ_n are given by

$$\alpha_n = \frac{a_n}{a_{n+1}}, \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \quad \gamma_n = \frac{a_{n-1}}{a_n} \frac{d_n^2}{d_{n-1}^2}, \tag{17}$$

with a_n and b_n being the first and second coefficients in the power expansion of P_n, i.e.,

$$P_n(s)_q = a_n x^n(s) + b_n x^{n-1}(s) + \cdots .$$

Substituting $s = a$ in (16) we obtain

$$\beta_n = \frac{x(a) P_n(a)_q - \alpha_n P_{n+1}(a)_q - \gamma_n P_{n-1}(a)_q}{P_n(a)_q}, \tag{18}$$

which is an alternative way to find the coefficient β_n.

Also we can use the following expression (see [26], p. 148):

$$\beta_n = \frac{[n]_q \tau_{n-1}(0)}{\tau'_{n-1}} - \frac{[n+1]_q \tau_n(0)}{\tau'_n} + c_3([n]_q + 1 - [n+1]_q).$$

To compute α_n (and β_n) we need the following formulas (see, e.g., [26], p. 147):

$$a_n = \frac{B_n A_{n,n}}{[n]_q!}, \qquad \frac{b_n}{a_n} = \frac{[n]_q \tau_{n-1}(0)}{\tau'_{n-1}} + c_3([n]_q - n). \tag{19}$$

The explicit expression of λ_n is (see Eq. (52) in [28], p. 232)

$$\begin{aligned} \lambda_n &= -\frac{A q^{\mu}}{c_1^2 (q^{1/2} - q^{-1/2})^4} [n]_q \, [s_1 + s_2 + s_3 + s_4 + 2\mu + n - 1]_q \\ &= -\frac{C\, q^{-n+1/2}}{c_1^2 (q^{1/2} - q^{-1/2})^2} \left(1 - q^n\right) \left(1 - q^{s_1+s_2+s_3+s_4+2\mu+n-1}\right), \end{aligned} \tag{20}$$

which can be obtained by equating the largest powers of q^s in (5).

From the Rodrigues formula (see [29] and [26], § 5.6) follows that

$$\frac{\Delta P_n(s-\frac{1}{2})_q}{\Delta x(s-\frac{1}{2})} = \frac{-\lambda_n B_n}{\widetilde{B}_{n-1}} \widetilde{P}_{n-1}(s)_q, \tag{21}$$

where $\widetilde{P}_{n-1}$ denotes the polynomial orthogonal with respect to the weight function $\widetilde{\rho}(s) = \rho_1(s-\frac{1}{2})$.

On the other hand, rewriting (3) as

$$\left(\sigma(s)\frac{\nabla}{\nabla x_1(s)} + \tau(s)I\right)\frac{\Delta}{\Delta x(s)} P_n(s)_q = -\lambda_n P_n(s)_q,$$

it can be replaced by the following two first-order difference equations

$$\frac{\Delta}{\Delta x(s)} P_n(s)_q = Q(s), \qquad \left(\sigma(s)\frac{\nabla}{\nabla x_1(s)} + \tau(s)I\right) Q(s) = -\lambda_n P_n(s)_q. \tag{22}$$

From the fact that $\frac{\Delta}{\Delta x(s)} P_n(s)_q$ is a polynomial of degree $n-1$ on $x(s+1/2)$ (see [9], § 3.1) follows that

$$\frac{\Delta}{\Delta x(s)} P_n(s)_q = C_n Q_{n-1}(s+\tfrac{1}{2}),$$

where C_n is a normalizing constant. A comparison with (21) implies that $Q(s)$ is the polynomial $\widetilde{P}_{n-1}$ orthogonal with respect to the function $\rho_1(s-\frac{1}{2})$ and $C_n = -\lambda_n B_n/\widetilde{B}_{n-1}$. Therefore, the second expression in (22) becomes

$$P_n(s)_q = \frac{B_n}{\widetilde{B}_{n-1}}\left(\sigma(s)\frac{\nabla}{\nabla x_1(s)} + \tau(s)I\right)\widetilde{P}_{n-1}(s+\tfrac{1}{2})_q. \tag{23}$$

The q-polynomials satisfy the following differentiation-type formula (see [29] and [26], § 5.6.1):

$$\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q \tau_n'}\left[\tau_n(s) P_n(s)_q - \frac{B_n}{B_{n+1}} P_{n+1}(s)_q\right]. \tag{24}$$

Then, using the explicit expression for the coefficient α_n, we obtain

$$\sigma(s)\frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q}\frac{\tau_n(s)}{\tau_n'} P_n(s)_q - \frac{\alpha_n \lambda_{2n}}{[2n]_q} P_{n+1}(s)_q. \tag{25}$$

From the above equation, in view of the identity

$$\Delta\frac{\nabla P_n(s)_q}{\nabla x(s)} = \frac{\Delta P_n(s)_q}{\Delta x(s)} - \frac{\nabla P_n(s)_q}{\nabla x(s)}$$

along with SODE (5), we find

$$\sigma(-s-\mu)\frac{\Delta P_n(s)_q}{\Delta x(s)} = \frac{\lambda_n}{[n]_q \tau_n'}\left[\big(\tau_n(s) - [n]_q \tau_n' \Delta x(s-\tfrac{1}{2})\big) P_n(s)_q - \frac{B_n}{B_{n+1}} P_{n+1}(s)_q\right]. \tag{26}$$

To conclude this section, we will introduce the following notation adopted from [9, 28].

First we define another q-analog of the Pocchammer symbols (see Eq. (3.11.1) in [9]):

$$(a|q)_k = \prod_{m=0}^{k-1} [a+m]_q = \frac{\widetilde{\Gamma}_q(a+k)}{\widetilde{\Gamma}_q(a)} = (-1)^k (q^a;q)_k (q^{1/2} - q^{-1/2})^{-k} q^{-\frac{k}{4}(k-1) - \frac{ka}{2}}, \tag{27}$$

where $\widetilde{\Gamma}_q(x)$ is the q-analog of the Gamma function introduced in [9] (see Eq. (3.2.24) therein) and related to the classical q-Gamma function Γ_q by the formula

$$\widetilde{\Gamma}_q(s) = q^{-\frac{(s-1)(s-2)}{4}} \Gamma_q(s) = q^{-\frac{(s-1)(s-2)}{4}} (1-q)^{1-s} \frac{(q;q)_\infty}{(q^s;q)_\infty}, \quad 0 < q < 1.$$

Next we define the q-hypergeometric function ${}_r\mathrm{F}_p(\cdot|q,z)$

$${}_r\mathrm{F}_p \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_p \end{matrix} \,\middle|\, q\,,\, z \right) = \sum_{k=0}^{\infty} \frac{(a_1|q)_k (a_2|q)_k \cdots (a_r|q)_k}{(b_1|q)_k (b_2|q)_k \cdots (b_p|q)_k} \frac{z^k}{(1|q)_k} \left[\varkappa_q^{-k} q^{\frac{1}{4}k(k-1)} \right]^{p-r+1}, \tag{28}$$

where, as before, $\varkappa_q = q^{1/2} - q^{-1/2}$ and $(a|q)_k$ are given by (27). Notice that

$$\lim_{q\to 1} {}_r\mathrm{F}_p \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_p \end{matrix} \,\middle|\, q\,,\, z\, \varkappa_q^{p-r+1} \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_p)_k} \frac{z^k}{k!} = {}_r\mathrm{F}_p \left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_p \end{matrix} \,\middle|\, z \right)$$

and

$${}_{p+1}\mathrm{F}_p \left(\begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, b_2, \dots, b_p \end{matrix} \,\middle|\, q\,,\, t \right) \Bigg|_{t\,=\,t_0} = {}_{p+1}\varphi_p \left(\begin{matrix} q^{a_1}, q^{a_2}, \dots, q^{a_{p+1}} \\ q^{b_1}, q^{b_2}, \dots, q^{b_p} \end{matrix} \,\middle|\, q\,,\, z \right), \tag{29}$$

where $t_0 = z\, q^{\frac{1}{2}\left(\sum_{i=1}^{p+1} a_i - \sum_{i=1}^{p} b_i - 1\right)}$.

With the above notation, the polynomial solutions of (3) (see Eq. (49) in [28], p. 232) read

$$\begin{aligned} P_n(s)_q = B_n \left(\frac{A}{c_1 q^{-\frac{\mu}{2}} \varkappa_q^2} \right)^n (s_1+s_2+\mu|q)_n (s_1+s_3+\mu|q)_n \\ \times (s_1+s_4+\mu|q)_n \; {}_4\mathrm{F}_3 \left(\begin{matrix} -n, 2\mu+n-1+\sum_{i=1}^{4} s_i, s_1-s, s_1+s+\mu \\ s_1+s_2+\mu, s_1+s_3+\mu, s_1+s_4+\mu \end{matrix} \,\middle|\, q\,,\, 1 \right). \end{aligned} \tag{30}$$

2.1. The q-Racah Polynomials

Here we consider the q-Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ on the lattice $x(s) = [s]_q[s+1]_q$ introduced in [9, 18, 30].

For this lattice, one has

$$c_1 = q^{\frac{1}{2}} \varkappa_q^{-2}, \quad \mu = 1, \quad c_3 = -(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \varkappa_q^{-2}. \tag{31}$$

We choose σ in (11) as follows:

$$\sigma(s) = -\frac{q^{-2s}}{\varkappa_q^4 q^{\frac{\alpha+\beta}{2}}}(q^s - q^a)(q^s - q^{-b})(q^s - q^{\beta-a})(q^s - q^{b+\alpha}) = [s-a]_q[s+b]_q[s+a-\beta]_q[b+\alpha-s]_q,$$

i.e.,

$$s_1 = a, s_2 = -b, s_3 = \beta - a, s_4 = b+\alpha, \quad C = -q^{-\frac{1}{2}(\alpha+\beta)}\varkappa_q^{-4}, \quad A = -1,$$

and let $B_n = (-1)^n/[n]_q!$. Here, as before, $\varkappa_q = q^{1/2} - q^{-1/2}$. Now from (20) we find

$$\lambda_n = q^{-\frac{1}{2}(\alpha+\beta+2n+1)}\varkappa_q^{-2}(1-q^n)(1-q^{\alpha+\beta+n+1}) = [n]_q[n+\alpha+\beta+1]_q.$$

To obtain $\tau_n(s)$ we use (9).

In this case, $x_n(s) = [s+n/2]_q[s+n/2+1]_q$; then, choosing $s_n^\star = -n/2$, we get

$$\tau_n(s) = \tau_n' x_n(s) + \tau_n(0), \quad \tau_n' = -[2n+\alpha+\beta+2]_q, \quad \tau_n(0) = \sigma(-n/2-1) - \sigma(-n/2). \tag{32}$$

Taking into account that $\tau(s) = \tau_0(s)$, we obtain the corresponding function $\tau(s)$,

$$\tau(s) = -[2+\alpha+\beta]_q x(s) + \sigma(-1) - \sigma(0).$$

2.1.1. The Orthogonality and the Norm d_n^2

A solution to the Pearson-type difference equation (8) reads

$$\rho(s) = \frac{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s)}.$$

Since $\sigma(a)\rho(a) = \sigma(b)\rho(b) = 0$, the q-Racah polynomials satisfy the orthogonality relation

$$\sum_{s=a}^{b-1} u_n^{\alpha,\beta}(x(s),a,b)_q u_m^{\alpha,\beta}(x(s),a,b)_q \rho(s)[2s+1]_q = 0, \qquad n \neq m,$$

with the restrictions $-\frac{1}{2} < a \leq b-1,\ \alpha > -1$, and $-1 < \beta < 2a+1$.

Let us now compute the square of the norm d_n^2.

From (7) and (15) follow

$$\rho_n(s) = \frac{\widetilde{\Gamma}_q(s+n+a+1)\widetilde{\Gamma}_q(s+n-a+\beta+1)\widetilde{\Gamma}_q(s+n+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s-n)}$$

and

$$A_{n,n} = [n]_q!(-1)^n\frac{\widetilde{\Gamma}_q(\alpha+\beta+2n+1)}{\widetilde{\Gamma}_q(\alpha+\beta+n+1)} \quad \Rightarrow \quad \Lambda_n := (-1)^n A_{n,n}B_n^2 = \frac{\widetilde{\Gamma}_q(\alpha+\beta+2n+1)}{[n]_q!\widetilde{\Gamma}_q(\alpha+\beta+n+1)}.$$

Taking into account that $\nabla x_{n+1}(s) = [2s+n+1]_q$, in view of (14) and the identity

$$\widetilde{\Gamma}_q(A-s) = \frac{\widetilde{\Gamma}_q(A)(-1)^s}{(1-A|q)_s}, \tag{33}$$

we have

$$d_n^2 = \Lambda_n \sum_{s=a}^{b-n-1} \frac{\widetilde{\Gamma}_q(s+n+a+1)\widetilde{\Gamma}_q(s+n-a+\beta+1)\widetilde{\Gamma}_q(s+n+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s-n)[2s+n+1]_q^{-1}}$$

$$= \Lambda_n \sum_{s=0}^{b-a-n-1} \frac{\widetilde{\Gamma}_q(s+n+2a+1)\widetilde{\Gamma}_q(s+n+\beta+1)\widetilde{\Gamma}_q(s+n+\alpha+b+a+1)\widetilde{\Gamma}_q(b-a+\alpha-s)}{\widetilde{\Gamma}_q(s+1)\widetilde{\Gamma}_q(s+b+a+1)\widetilde{\Gamma}_q(s+2a-\beta+1)\widetilde{\Gamma}_q(b-a-s-n)[2s+2a+n+1]_q^{-1}}$$

$$= \frac{\widetilde{\Gamma}_q(\alpha+\beta+2n+1)\widetilde{\Gamma}_q(2a+n+1)\widetilde{\Gamma}_q(n+\beta+1)\widetilde{\Gamma}_q(a+b+n+\alpha+1)\widetilde{\Gamma}_q(b+\alpha-a)}{[n]_q!\widetilde{\Gamma}_q(\alpha+\beta+n+1)\widetilde{\Gamma}_q(a+b+1)\widetilde{\Gamma}_q(2a-\beta+1)\widetilde{\Gamma}_q(b-a-n)}$$

$$\times \sum_{s=0}^{b-a-n-1} \frac{(n+2a+1, n+\beta+1, n+a+\alpha+b+1, 1-b+a+n|q)_s}{(1, a+b+1, 2a-\beta+1, 1-b+a-\alpha|q)_s}[2s+2a+n+1]_q.$$

In the following, we denote by S_n the sum in the last expression. If we now use that

$$(a|q)_n = (-1)^n (q^a;q)_n q^{-\frac{n}{4}(n+2a-1)} \varkappa_q^{-n},$$

as well as the identity

$$[2s+2a+n+1]_q = q^{-s}[2a+n+1]_q \frac{(q^{a+\frac{n+1}{2}+1};q)(-q^{a+\frac{n+1}{2}+1};q)_s}{(q^{a+\frac{n+1}{2}};q)(-q^{a+\frac{n+1}{2}};q)},$$

we obtain

$$S_n = \sum_{s=0}^{b-a-n-1} \frac{(q^{2a+n+1}, q^{n+\beta+1}, q^{n+\alpha+b+a+1}, q^{1-b+a+n}, q^{\frac{1}{2}(2a+n+3)}, -q^{\frac{1}{2}(2a+n+3)};q)_s}{(q, q^{a+b+1}, q^{2a-\beta+1}, q^{1-b-\alpha+a}, q^{\frac{1}{2}(2a+n+1)}, -q^{\frac{1}{2}(2a+n+1)};q)_s [2a+n+1]_q^{-1}} q^{-s(1+2n+\beta+\alpha)}$$

$$= [2a+n+1]_q \, {}_6\phi_5\left(\left. \begin{matrix} q^{2a+n+1}, q^{n+\beta+1}, q^{n+a+\alpha+b+1}, q^{1-b+a+n}, q^{\frac{1}{2}(2a+n+3)}, -q^{\frac{1}{2}(2a+n+3)} \\ q^{a+b+1}, q^{2a-\beta+1}, q^{1-b-\alpha+a}, q^{\frac{1}{2}(2a+n+1)}, -q^{\frac{1}{2}(2a+n+1)} \end{matrix} \right| q, q^{-1-2n-\beta-\alpha} \right).$$

But the above ${}_6\phi_5$ series is a very-well-posed ${}_6\phi_5$ basic series and, therefore, by using the summation formula (see Eq. (II.21) in [31], p. 238)

$${}_6\varphi_5\left(\left. \begin{matrix} a, \; qa^{1/2}, \; -qa^{1/2}, \; b, \; c, \; q^{-k} \\ a^{1/2}, \; -a^{1/2}, aq/b, \; aq/c, aq^{k+1} \end{matrix} \right| q, \frac{aq^{k+1}}{bc} \right) = \frac{(aq, aq/bc;q)_k}{(aq/b, aq/c;q)_k},$$

with $k = b-a-n-1$, $a = q^{2a+n+1}$, $b = q^{n+\beta+1}$, and $c = q^{n+a+\alpha+b+1}$, we obtain

$$\begin{aligned} S_n &= [2a+n+1]_q \frac{(q^{2a+n+2}, q^{-n+a-b-\alpha-\beta};q)_{b-a-n-1}}{(q^{2a-\beta+1}, q^{a-b-\alpha+1};q)_{b-a-n-1}} \\ &= [2a+n+1]_q \frac{(2a+n+2|q)_{b-a-n-1}(-n+a-b-\alpha-\beta|q)_{b-a-n-1}}{(2a-\beta+1|q)_{b-a-n-1}(a-b-\alpha+1|q)_{b-a-n-1}}. \end{aligned}$$

Finally, in view of (33) and (27), we arrive at

$$S_n = [2a+n+1]_q \frac{\widetilde{\Gamma}_q(a+b+1)\widetilde{\Gamma}_q(2a-\beta+1)\widetilde{\Gamma}_q(b-a+\alpha+\beta+n+1)\widetilde{\Gamma}_q(\alpha+n+1)}{\widetilde{\Gamma}_q(n+2a+2)\widetilde{\Gamma}_q(b+a-\beta-n)\widetilde{\Gamma}_q(\alpha+\beta+2n+2)\widetilde{\Gamma}_q(b-a+\alpha)};$$

thus

$$\begin{aligned} d_n^2 &= \frac{\widetilde{\Gamma}_q(\alpha+\beta+2n+1)\widetilde{\Gamma}_q(2a+n+1)\widetilde{\Gamma}_q(n+\beta+1)\widetilde{\Gamma}_q(a+b+n+\alpha+1)\widetilde{\Gamma}_q(b+\alpha-a)}{[n]_q!\widetilde{\Gamma}_q(\alpha+\beta+n+1)\widetilde{\Gamma}_q(a+b+1)\widetilde{\Gamma}_q(2a-\beta+1)\widetilde{\Gamma}_q(b-a-n)} S_n \\ &= \frac{\widetilde{\Gamma}_q(\alpha+n+1)\widetilde{\Gamma}_q(\beta+n+1)\widetilde{\Gamma}_q(b-a+\alpha+\beta+n+1)\widetilde{\Gamma}_q(a+b+\alpha+n+1)}{[\alpha+\beta+2n+1]_q\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(\alpha+\beta+n+1)\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(a+b-\beta-n)}. \end{aligned}$$

2.1.2. The Hypergeometric Representation

From formulas (12) and (30) the following two equivalent hypergeometric representations hold:

$$\begin{aligned} u_n^{\alpha,\beta}(x(s),a,b)_q =& \frac{q^{-\frac{n}{2}(2a+\alpha+\beta+n+1)}(q^{a-b+1};q)_n(q^{\beta+1};q)_n(q^{a+b+\alpha+1};q)_n}{\varkappa_q^{2n}(q;q)_n} \\ & \times {}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{\alpha+\beta+n+1}, q^{a-s}, q^{a+s+1} \\ q^{a-b+1}, q^{\beta+1}, q^{a+b+\alpha+1} \end{matrix} \,\middle|\, q\,,\,q\right) \end{aligned} \tag{34}$$

and

$$\begin{aligned} u_n^{\alpha,\beta}(x(s),a,b)_q =& \frac{(a-b+1|q)_n(\beta+1|q)_n(a+b+\alpha+1|q)_n}{[n]_q!} \\ & \times {}_4\mathrm{F}_3\left(\begin{matrix} -n, \alpha+\beta+n+1, a-s, a+s+1 \\ a-b+1, \beta+1, a+b+\alpha+1 \end{matrix} \,\middle|\, q\,,\,1\right). \end{aligned} \tag{35}$$

In view of the Sears transformation formula (see Eq. (III.15) in [31]), we obtain the equivalent formulas

$$\begin{aligned} u_n^{\alpha,\beta}(x(s),a,b)_q =& \frac{q^{-\frac{n}{2}(-2b+\alpha+\beta+n+1)}(q^{a-b+1};q)_n(q^{\alpha+1};q)_n(q^{\beta-a-b+1};q)_n}{\varkappa_q^{2n}(q;q)_n} \\ & \times {}_4\varphi_3\left(\begin{matrix} q^{-n}, q^{\alpha+\beta+n+1}, q^{-b-s}, q^{-b+s+1} \\ q^{a-b+1}, q^{\alpha+1}, q^{-a-b+\beta+1} \end{matrix} \,\middle|\, q\,,\,q\right) \end{aligned} \tag{36}$$

and

$$\begin{aligned} u_n^{\alpha,\beta}(x(s),a,b)_q =& \frac{(a-b+1|q)_n(\alpha+1|q)_n(-a-b+\beta+1|q)_n}{[n]_q!} \\ & \times {}_4\mathrm{F}_3\left(\begin{matrix} -n, \alpha+\beta+n+1, -b-s, -b+s+1 \\ a-b+1, \alpha+1, -a-b+\beta+1 \end{matrix} \,\middle|\, q\,,\,1\right). \end{aligned} \tag{37}$$

Remark: From the above formulas follow that the polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ are multiples of the standard q-Racah polynomials $R_n(\mu(q^{b+s});q^\alpha,q^\beta,q^{a-b},q^{-a-b}|q)$.

From the above hypergeometric representations also follow the values

$$u_n^{\alpha,\beta}(x(a),a,b)_q = \frac{(a-b+1|q)_n(\beta+1|q)_n(a+b+\alpha+1|q)_n}{[n]_q!} = \frac{(q^{a-b+1};q)_n(q^{\beta+1};q)_n(q^{a+b+\alpha+1};q)_n}{q^{\frac{n}{2}(2a+\alpha+\beta+n+1)}\varkappa_q^{2n}(q;q)_n} \tag{38}$$

and

$$u_n^{\alpha,\beta}(x(b-1),a,b)_q = \frac{(a-b+1|q)_n(\alpha+1|q)_n(-a-b+\beta+1|q)_n}{[n]_q!} = \frac{(q^{a-b+1};q)_n(q^{\alpha+1};q)_n(q^{\beta-a-b+1};q)_n}{q^{\frac{n}{2}(-2b+\alpha+\beta+n+1)}\varkappa_q^{2n}(q;q)_n}. \tag{39}$$

Formula (10) leads to the following explicit formula[2]:

$$u_n^{\alpha,\beta}(x(s),a,b)_q = \frac{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s)}{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)} \times \sum_{k=0}^{n} \frac{(-1)^k[2s+2k-n+1]_q\widetilde{\Gamma}_q(s+k+a+1)\widetilde{\Gamma}_q(2s+k-n+1)}{\widetilde{\Gamma}_q(k+1)\widetilde{\Gamma}_q(n-k+1)\widetilde{\Gamma}_q(2s+k+2)\widetilde{\Gamma}_q(s-n+k-a+1)} \times \frac{\widetilde{\Gamma}_q(s+k-a+\beta+1)\widetilde{\Gamma}_q(s+k+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s+n-k)}{\widetilde{\Gamma}_q(s-n+k+b+1)\widetilde{\Gamma}_q(s-n+k+a-\beta+1)\widetilde{\Gamma}_q(b-s-k)}, \tag{40}$$

from which follow

$$u_n^{\alpha,\beta}(x(a),a,b)_q = \frac{(-1)^n\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(\beta+n+1)\widetilde{\Gamma}_q(b+a+\alpha+n+1)}{[n]!\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(\beta+1)\widetilde{\Gamma}_q(b+a+\alpha+1)}, \quad u_n^{\alpha,\beta}(x(b-1),a,b)_q = \frac{\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(\alpha+n+1)\widetilde{\Gamma}_q(b+a-\beta)}{[n]!\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(\alpha+1)\widetilde{\Gamma}_q(b+a-\beta-n)}, \tag{41}$$

which coincide with values (38) and (39) obtained before.

From the hypergeometric representation the following symmetry property follows:

$$u_n^{\alpha,\beta}(x(s),a,b)_q = u_n^{-b-a+\beta,b+a+\alpha}(x(s),a,b)_q.$$

Finally, notice that from (34) [or (36)] follows that $u_n^{\alpha,\beta}(x(s),a,b)_q$ is a polynomial of degree n on $x(s)=[s]_q[s+1]_q$. In fact,

$$(q^{a-s};q)_k(q^{a+s+1};q)_k = (-1)^k q^{k(a+\frac{k+1}{2})}\prod_{l=0}^{k-1}\left(\frac{x(s)-c_3}{c_1} - q^{-\frac{1}{2}}(q^{a+l+\frac{1}{2}}+q^{-a-l-\frac{1}{2}})\right),$$

where c_1 and c_3 are given by (31).

[2]Obviously formulas (34) and (36) also give equivalent explicit formulas.

2.1.3. Three-Term Recurrence Relation and Differentiation Formulas

To derive the coefficients of TTRR (16), we use (17) and (18).

In view of (19) and (17), we obtain

$$a_n = \frac{\widetilde{\Gamma}_q(\alpha+\beta+2n+1)}{[n]_q!\widetilde{\Gamma}_q(\alpha+\beta+n+1)}, \quad \alpha_n = \frac{[n+1]_q[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q}.$$

To find γ_n, we use (17)

$$\gamma_n = \frac{[a+b+\alpha+n]_q[a+b-\beta-n]_q[\alpha+n]_q[\beta+n]_q[b-a+\alpha+\beta+n]_q[b-a-n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}.$$

To compute β_n, we use (18)

$$\begin{aligned}\beta_n = & \ x(a) - \alpha_n \frac{u_{n+1}^{\alpha,\beta}(x(a),a,b)_q}{u_n^{\alpha,\beta}(x(a),a,b)_q} - \gamma_n \frac{u_{n-1}^{\alpha,\beta}(x(a),a,b)_q}{u_n^{\alpha,\beta}(x(a),a,b)_q} \\ = & \ [a]_q[a+1]_q - \frac{[\alpha+\beta+n+1]_q[a-b+n+1]_q[\beta+n+1]_q[a+b+\alpha+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q} \\ & + \frac{[\alpha+n]_q[b-a+\alpha+\beta+n]_q[a+b-\beta-n]_q[n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}.\end{aligned}$$

The differentiation formulas (21) and (23) yield

$$\frac{\Delta u_n^{\alpha,\beta}(x(s),a,b)_q}{\Delta x(s)} = [\alpha+\beta+n+1]_q u_{n-1}^{\alpha+1,\beta+1}(x(s+\tfrac{1}{2}),a+\tfrac{1}{2},b-\tfrac{1}{2})_q, \tag{42}$$

$$\begin{aligned}-[n]_q[2s+1]_q u_n^{\alpha,\beta}(x(s),a,b)_q = & \sigma(-s-1)u_{n-1}^{\alpha+1,\beta+1}(x(s+\tfrac{1}{2}),a+\tfrac{1}{2},b-\tfrac{1}{2})_q \\ & - \sigma(s)u_{n-1}^{\alpha+1,\beta+1}(x(s-\tfrac{1}{2}),a+\tfrac{1}{2},b-\tfrac{1}{2})_q,\end{aligned} \tag{43}$$

respectively.

Finally, formulas (24) [or (25)] and (26) lead to the differentiation formulas

$$\sigma(s)\frac{\nabla u_n^{\alpha,\beta}(x(s),a,b)_q}{[2s]_q} = -\frac{[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+2]_q}\left[\tau_n(s)u_n^{\alpha,\beta}(x(s),a,b)_q + [n+1]_q u_{n+1}^{\alpha,\beta}(x(s),a,b)_q\right] \tag{44}$$

and

$$\begin{aligned}\sigma(-s-1)\frac{\Delta u_n^{\alpha,\beta}(x(s),a,b)_q}{[2s+2]_q} = & -\frac{[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+2]_q} \\ \times & \left[(\tau_n(s)+[n]_q[\alpha+\beta+2n+2]_q[2s+1]_q)u_n^{\alpha,\beta}(x(s),a,b)_q + [n+1]_q u_{n+1}^{\alpha,\beta}(x(s),a,b)_q\right],\end{aligned} \tag{45}$$

where $\tau_n(s)$ is given by (32).

TABLE 1. Main Data of q-Racah Polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$.

$P_n(s)$	$u_n^{\alpha,\beta}(x(s),a,b)_q\,, \quad x(s)=[s]_q[s+1]_q$
(a,b)	$[a,b-1]$
$\rho(s)$	$\dfrac{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s)}$ $-\frac{1}{2}<a\le b-1, \alpha>-1, -1<\beta<2a+1$
$\sigma(s)$	$[s-a]_q[s+b]_q[s+a-\beta]_q[b+\alpha-s]_q$
$\sigma(-s-1)$	$[s+a+1]_q[b-s-1]_q[s-a+\beta+1]_q[b+\alpha+s+1]_q$
$\tau(s)$	$[\alpha+1]_q[a]_q[a-\beta]_q+[\beta+1]_q[b]_q[b+\alpha]_q-[\alpha+1]_q[\beta+1]_q-[\alpha+\beta+2]_q x(s)$
$\tau_n(s)$	$-[\alpha+\beta+2n+2]_q x(s+\frac{n}{2})+[a+\frac{n}{2}+1]_q[b-\frac{n}{2}-1]_q[\beta+\frac{n}{2}+1-a]_q[b+\alpha+\frac{n}{2}+1]_q$ $-[a+\frac{n}{2}]_q[b-\frac{n}{2}]_q[\beta+\frac{n}{2}-a]_q[b+\alpha+\frac{n}{2}]_q$
λ_n	$[n]_q[\alpha+\beta+n+1]_q$
B_n	$\dfrac{(-1)^n}{[n]_q!}$
d_n^2	$\dfrac{\widetilde{\Gamma}_q(\alpha+n+1)\widetilde{\Gamma}_q(\beta+n+1)\widetilde{\Gamma}_q(b-a+\alpha+\beta+n+1)\widetilde{\Gamma}_q(a+b+\alpha+n+1)}{[\alpha+\beta+2n+1]_q\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(\alpha+\beta+n+1)\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(a+b-\beta-n)}$
$\rho_n(s)$	$\dfrac{\widetilde{\Gamma}_q(s+n+a+1)\widetilde{\Gamma}_q(s+n-a+\beta+1)\widetilde{\Gamma}_q(s+n+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)}{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(b-s-n)}$
a_n	$\dfrac{\widetilde{\Gamma}_q[\alpha+\beta+2n+1]_q}{[n]_q!\widetilde{\Gamma}_q[\alpha+\beta+n+1]_q}$
α_n	$\dfrac{[n+1]_q[\alpha+\beta+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q}$
β_n	$[a]_q[a+1]_q-\dfrac{[\alpha+\beta+n+1]_q[a-b+n+1]_q[\beta+n+1]_q[a+b+\alpha+n+1]_q}{[\alpha+\beta+2n+1]_q[\alpha+\beta+2n+2]_q}$ $+\dfrac{[\alpha+n]_q[b-a+\alpha+\beta+n]_q[a+b-\beta-n]_q[n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}$
γ_n	$\dfrac{[a+b+\alpha+n]_q[a+b-\beta-n]_q[\alpha+n]_q[\beta+n]_q[b-a+\alpha+\beta+n]_q[b-a-n]_q}{[\alpha+\beta+2n]_q[\alpha+\beta+2n+1]_q}$

2.1.4. The Duality of Racah Polynomials

In this section, we discuss the duality property of the q-Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$.

We will follow [9] (see pp. 38, 39 therein).

First of all, notice that the orthogonal relation (13) for the Racah polynomials can be written in the form

$$\sum_{t=0}^{N-1} C_{tn}C_{tm} = \delta_{n,m}, \quad C_{tn} = \frac{u_n^{\alpha,\beta}(x(t+a),a,b)_q\sqrt{\rho(t+a)\Delta x(t+a-1/2)}}{d_n}, \quad N=b-a,$$

where $\rho(s)$ and d_n are the weight function and the norm of q-Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$, respectively. The above relation can be understood as the orthogonality property of the matrix $C=(C_{tn})_{t,n=0}^{N-1}$ by its first index. If we now use the orthogonality of C by the second index, we get

$$\sum_{n=0}^{N-1} C_{tn}C_{t'n} = \delta_{t,t'}, \quad N=b-a,$$

that leads to the dual orthogonality relation for the q-Racah polynomials

$$\sum_{n=0}^{N-1} u_n^{\alpha,\beta}(x(s),a,b)_q u_n^{\alpha,\beta}(x(s'),a,b)_q \frac{1}{d_n^2} = \frac{1}{\rho(s)\Delta x(s-1/2)}\delta_{s,s'}. \tag{46}$$

The next step is to identify the functions $u_n^{\alpha,\beta}(x(s),a,b)_q$ as polynomials on some lattice $x(n)$.

Before starting, let us mention that from the representation (34) and the identity

$$(q^{-n};q)_k(q^{\alpha+\beta+n+1};q)_k = \prod_{l=0}^{k-1}\left(1+q^{\alpha+\beta+2l+1} - q^{\frac{\alpha+\beta+1}{2}+l}\left(\varkappa_q^2 x(t) + q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right)\right),$$

where

$$x(t) = [t]_q[t+1]_q = [n+\frac{\alpha+\beta}{2}]_q[n+\frac{\alpha+\beta}{2}+1]_q,$$

follows that $u_n^{\alpha,\beta}(x(s),a,b)_q$ also constitutes a polynomial of degree $s-a$ (for $s=a,a+1,\dots,b-a-1$) on $x(t)$ with $t=n+\frac{\alpha+\beta}{2}$.

Let us now define the polynomials [compare with the definition of the Racah polynomials (35)]

$$\begin{aligned}\mathrm{u}_k^{\alpha',\beta'}(x(t),a',b')_q = &\frac{(-1)^k\widetilde{\Gamma}_q(b'-a')\widetilde{\Gamma}_q(\beta'+k+1)\widetilde{\Gamma}_q(b'+a'+\alpha'+k+1)}{[k]!\widetilde{\Gamma}_q(b'-a'-k)\widetilde{\Gamma}_q(\beta'+1)\widetilde{\Gamma}_q(b'+a'+\alpha'+1)} \\ &\times {}_4\mathrm{F}_3\left(\begin{matrix}-k,\alpha'+\beta'+k+1,a'-t,a'+t+1\\ a'-b'+1,\beta'+1,a'+b'+\alpha'+1\end{matrix}\,\middle|\, q\,,1\right),\end{aligned} \tag{47}$$

where

$$k=s-a, \quad t=n+\frac{\alpha+\beta}{2}, \quad a'=\frac{\alpha+\beta}{2}, \quad b'=b-a+\frac{\alpha+\beta}{2}, \quad \alpha'=2a-\beta, \quad \beta'=\beta. \tag{48}$$

Obviously they are polynomials of degree $k = s - a$ on the lattice $x(t)$ that satisfy the orthogonality property

$$\sum_{t=a'}^{b'-1} \mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q \mathrm{u}_m^{\alpha',\beta'}(x(t), a', b')_q \rho'(t) \Delta x(t - 1/2) = (d'_k)^2 \delta_{k,m}, \tag{49}$$

where $\rho'(t)$ and d'_k are the weight function ρ and the norm d_n is given in Table 1 with the corresponding change of a, b, α, β, s, n by a', b', α', β', t, k.

Furthermore, with the above choice (48) of the parameters of $\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q$, the hypergeometric function ${}_4\mathrm{F}_3$ in (47) coincides with the function ${}_4\mathrm{F}_3$ in (35) and, therefore, the following relation between the polynomials $\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')$ and $u_n^{\alpha,\beta}(x(s), a, b)_q$ holds:

$$\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q = \mathcal{A}(\alpha, \beta, a, b, n, s) u_n^{\alpha,\beta}(x(s), a, b)_q, \tag{50}$$

where

$$\mathcal{A}(\alpha, \beta, a, b, n, s) = \frac{(-1)^{s-a+n} \widetilde{\Gamma}_q(b - a - n) \widetilde{\Gamma}_q(s - a + \beta + 1) \widetilde{\Gamma}_q(b + \alpha + s + 1) \widetilde{\Gamma}_q(n + 1)}{\widetilde{\Gamma}_q(b - s) \widetilde{\Gamma}_q(n + \beta + 1) \widetilde{\Gamma}_q(b + a + \alpha + n + 1) \widetilde{\Gamma}_q(s - a + 1)}.$$

If we now substitute (50) in (49) and make the change (48), then (49) converts into relation (46), i.e., the polynomial set $\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ defined by (47) [or (50)] is the dual set associated to the Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$.

To conclude this study, let us show that TTRR (16) of the polynomials $\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ is SODE (4) of the polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$, whereas SODE (4) of $\mathrm{u}_k^{\alpha',\beta'}(x(t), a', b')_q$ converts into TTRR (16) of $u_n^{\alpha,\beta}(x(s), a, b)_q$ and vice versa.

Let us denote by $\varsigma(t)$ the σ function of the polynomial $\mathrm{u}_k^{\alpha',\beta'}$; then

$$\varsigma(t) = [t - a']_q [t + b']_q [t + a' - \beta']_q [b' + \alpha' - t]_q = [n]_q [n + b - a + \alpha + \beta]_q [n + \alpha]_q [b + a - n - \beta]_q$$

and, therefore,

$$\varsigma(-t - 1) = [\alpha + \beta + n + 1]_q [b + a + \alpha + n + 1]_q [b - a - n - 1]_q [n + \beta + 1]_q,$$

$$\lambda_k = [k]_q [\alpha' + \beta' + k + 1]_q = [s - a]_q [s + a + 1]_q.$$

For the coefficients α'_k, β'_k, and γ'_k of TTRR for the polynomials $\mathrm{u}_k^{\alpha',\beta'}$, we have

$$\alpha'_k = \frac{[k + 1]_q [\alpha' + \beta' + k + 1]_q}{[\alpha' + \beta' + 2k + 1]_q [\alpha' + \beta' + 2k + 2]_q} = \frac{[s - a + 1]_q [s + a + 1]_q}{[2s + 1]_q [2s + 2]_q},$$

$$\gamma'_k = \frac{[b + \alpha + s]_q [b + \alpha - s]_q [s + a - \beta]_q [s - a + \beta]_q [b + s]_q [b - s]_q}{[2s + 1]_q [2s]_q},$$

and

$$\beta'_k = [n + \frac{\alpha + \beta}{2}]_q [n + \frac{\alpha + \beta}{2} + 1]_q + \frac{\sigma(-s - 1)}{[2s + 1]_q [2s + 2]_q} + \frac{\sigma(s)}{[2s + 1]_q [2s]_q}.$$

Also we have $\Delta x(t) = [2t + 2]_q = [2n + \alpha + \beta + 2]_q$ and $x(s) = [s]_q [s + 1]_q = [k + a]_q [k + a + 1]_q$.

We show that SODE of the Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ is TTRR of the polynomials $\mathrm{u}_k^{\alpha',\beta'}(x(t),a',b')_q$.

First, we substitute relation (50) in SODE (4) of the polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ and use that $u_n^{\alpha,\beta}(x(s\pm 1),a,b)_q$ is proportional to $\mathrm{u}_{k\pm 1}^{\alpha',\beta'}(x(t),a',b')_q$ [see (50)].

After some simplification, in view of the last formulas, we obtain

$$\alpha'_k \mathrm{u}_{k+1}^{\alpha',\beta'}(x(t),a',b')_q + \left(\beta'_k - [n]_q[\alpha+\beta+n+1]_q - [\tfrac{\alpha+\beta}{2}]_q[\tfrac{\alpha+\beta}{2}+1]_q\right) \mathrm{u}_k^{\alpha',\beta'}(x(t),a',b')_q + \gamma'_k \mathrm{u}_{k-1}^{\alpha',\beta'}(x(t),a',b')_q = 0,$$

but

$$[n]_q[\alpha+\beta+n+1]_q + [\tfrac{\alpha+\beta}{2}]_q[\tfrac{\alpha+\beta}{2}+1]_q = [n+\tfrac{\alpha+\beta}{2}]_q[n+\tfrac{\alpha+\beta}{2}+1]_q = x(t),$$

i.e., we obtain TTRR of the polynomials $\mathrm{u}_k^{\alpha',\beta'}(x(t),a',b')_q$.

If we now substitute (50) in TTRR (16) for the Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ and use that $u_{n\pm 1}^{\alpha,\beta}(x(s),a,b)_q \sim \mathrm{u}_k^{\alpha',\beta'}(x(t\pm 1),a',b')_q$, then we obtain SODE

$$\frac{\varsigma(-t-1)}{\Delta x(t)\Delta x(t-\frac12)} \mathrm{u}_k^{\alpha',\beta'}(x(t+1),a',b')_q + \frac{\varsigma(t)}{\nabla x(t)\Delta x(t-\frac12)} \mathrm{u}_k^{\alpha',\beta'}(x(t-1),a',b')_q$$
$$- \left[\frac{\varsigma(-t-1)}{\Delta x(t)\Delta x(t-\frac12)} + \frac{\varsigma(t)}{\nabla x(t)\Delta x(t-\frac12)} + [a]_q[a+1]_q - [k+a]_q[k+a+1]_q\right] \mathrm{u}_k^{\alpha',\beta'}(x(t),a',b')_q = 0.$$

That is SODE (4) of $\mathrm{u}_k^{\alpha',\beta'}(x(t),a',b')_q$ since

$$[a]_q[a+1]_q - [k+a]_q[k+a+1]_q = -[k]_q[k+2a+1]_q = -[k]_q[k+\alpha'+\beta'+1]_q = -\lambda_k.$$

2.2. The q-Racah Polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$

There exists another possibility to define the q-Racah polynomials as was suggested in [9, 18].

It corresponds to the function

$$\sigma(s) = [s-a]_q[s+b]_q[s-a+\beta]_q[b+\alpha+s]_q,$$

i.e., $A=1$, $s_1=a$, $s_2=-b$, $s_3=a-\beta$, $s_4=-b-\alpha$.

With this choice, we obtain a new family of polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ that is orthogonal with respect to the weight function

$$\rho(s) = \frac{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s+a-\beta+1)}{\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b-s)}.$$

All their characteristics can be obtained exactly in the same way as before. Moreover, they can also be obtained from the corresponding characteristics of the polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ by changing $\alpha\to -2b-\alpha$, $\beta\to 2a-\beta$ and using the properties of the functions $\widetilde{\Gamma}_q(s)$, $\Gamma_q(s)$, $(a|q)_n$, and $(a;q)_n$. We summarize the main data of the polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ in Table 2.

TABLE 2. Main Data of q-Racah Polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$.

$P_n(s)$	$\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q\,,\quad x(s)=[s]_q[s+1]_q$
(a,b)	$[a,b-1]$
$\rho(s)$	$\dfrac{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s+a-\beta+1)}{\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s)\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b-s)}$ $-\frac{1}{2}<a\le b-1,\alpha>-1,-1<\beta<2a+1$
$\sigma(s)$	$[s-a]_q[s+b]_q[s-a+\beta]_q[b+\alpha+s]_q$
$\sigma(-s-1)$	$[s+a+1]_q[b-s-1]_q[s+a-\beta+1]_q[b+\alpha-s-1]_q$
$\tau(s)$	$[2a-\beta+1]_q[b]_q[b+\alpha]_q-[2b+\alpha-1]_q[a]_q[a-\beta]_q-[2b+\alpha-1]_q[2a-\beta+1]_q$ $-[2b-2a+\alpha+\beta-2]_q x(s)$
$\tau_n(s)$	$-[2b-2a+\alpha+\beta-2n-2]_q x(s+\frac{n}{2})+[a+\frac{n}{2}+1]_q[b-\frac{n}{2}-1]_q[a+\frac{n}{2}+1-\beta]_q[b-\frac{n}{2}+\alpha-1]_q$ $-[a+\frac{n}{2}]_q[b-\frac{n}{2}]_q[a+\frac{n}{2}-\beta]_q[b-\frac{n}{2}+\alpha]_q$
λ_n	$[n]_q[2b-2a+\alpha+\beta-n-1]_q$
B_n	$\dfrac{1}{[n]_q!}$
d_n^2	$\dfrac{\widetilde{\Gamma}_q(2a+n-\beta+1)\widetilde{\Gamma}_q(2b-2a+\alpha+\beta-n)[2b-2a-2n-1+\alpha+\beta]_q^{-1}}{\widetilde{\Gamma}_q(n+1)\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(b-a-n+\alpha)\widetilde{\Gamma}_q(b-a+\beta-n)\widetilde{\Gamma}_q(2b+\alpha-n)\widetilde{\Gamma}_q(b-a+\alpha+\beta-n)}$
$\rho_n(s)$	$\dfrac{\widetilde{\Gamma}_q(s+a+n+1)\widetilde{\Gamma}_q(s+a+n-\beta+1)}{\widetilde{\Gamma}_q(s+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s-n)\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b-s-n)}$
a_n	$\dfrac{(-1)^n\widetilde{\Gamma}_q[2b-2a+\alpha+\beta-n]_q}{[n]_q!\widetilde{\Gamma}_q[2b-2a+\alpha+\beta-2n]_q}$
α_n	$-\dfrac{[n+1]_q[2b-2a+\alpha+\beta-n-1]_q}{[2b-2a+\alpha+\beta-2n-1]_q[2b-2a+\alpha+\beta-2n-2]_q}$
β_n	$[a]_q[a+1]_q+\dfrac{[2b-2a+\alpha+\beta-n-1]_q[a-b+n+1]_q[2a-\beta+n+1]_q[a-b-\alpha+n+1]_q}{[2b-2a+\alpha+\beta-2n-1]_q[2b-2a+\alpha+\beta-2n-2]_q}$ $+\dfrac{[2b+\alpha-n]_q[b-a+\alpha+\beta-n]_q[b-a+\beta-n]_q[n]_q}{[2b-2a+\alpha+\beta-2n-1]_q[2b-2a+\alpha+\beta-2n]_q}$
γ_n	$-\dfrac{[2a-\beta+n]_q[b-a-n]_q[b-a-n+\alpha]_q[b-a-n+\beta]_q[2b+\alpha-n]_q[b-a+\alpha+\beta-n]_q}{[2b-2a+\alpha+\beta-2n-1]_q[2b-2a+\alpha+\beta-2n]_q}$

2.2.1. The Hypergeometric Representation

For the $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ polynomials, we have the following hypergeometric representation:

$$\begin{aligned}\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q =&\frac{q^{-\frac{n}{2}(4a-2b-\alpha-\beta+n+1)}(q^{a-b+1};q)_n(q^{2a-\beta+1};q)_n(q^{a-b-\alpha+1};q)_n}{\varkappa_q^{2n}(q;q)_n}\\ &\times\ {}_4\varphi_3\left(\begin{array}{c} q^{-n},q^{2a-2b-\alpha-\beta+n+1},q^{a-s},q^{a+s+1}\\ q^{a-b+1},q^{2a-\beta+1},q^{a-b-\alpha+1}\end{array}\,\middle|\, q\,,\,q\right),\end{aligned} \tag{51}$$

or, in terms of the q-hypergeometric series (28),

$$\begin{aligned}\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q =&\frac{(a-b+1|q)_n(2a-\beta+1|q)_n(a-b-\alpha+1|q)_n}{[n]_q!}\\ &\times_4\mathrm{F}_3\left(\begin{array}{c} -n,2a-2b-\alpha-\beta+n+1,a-s,a+s+1\\ a-b+1,2a-\beta+1,a-b-\alpha+1\end{array}\,\middle|\, q\,,\,1\right).\end{aligned} \tag{52}$$

Using the Sears transformation formula (see Eq. (III.15) in [31]) we obtain the equivalent representation formulas:

$$\begin{aligned}\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q =&\frac{q^{-\frac{n}{2}(2a-4b-\alpha-\beta+n+1)}(q^{a-b+1};q)_n(q^{-2b-\alpha+1};q)_n(q^{-\beta+a-b+1};q)_n}{\varkappa_q^{2n}(q;q)_n}\\ &times_4\varphi_3\left(\begin{array}{c} q^{-n},q^{2a-2b-\alpha-\beta+n+1},q^{-b-s},q^{-b+s+1}\\ q^{a-b+1},q^{-2b-\alpha+1},q^{a-b-\beta+1}\end{array}\,\middle|\, q\,,\,q\right)\end{aligned} \tag{53}$$

and

$$\begin{aligned}\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q =&\frac{(a-b+1|q)_n(-2b-\alpha+1|q)_n(a-b-\beta+1|q)_n}{[n]_q!}\\ &\times\ {}_4\mathrm{F}_3\left(\begin{array}{c} -n,2a-2b-\alpha-\beta+n+1,-b-s,-b+s+1\\ a-b+1,-2b-\alpha+1,a-b-\beta+1\end{array}\,\middle|\, q\,,\,1\right).\end{aligned} \tag{54}$$

Remark: From the above formulas follows that the polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ are multiples of the standard q-Racah polynomials $R_n(\mu(q^{a-s});q^{a-b-\alpha},q^{a-b-\beta},q^{a-b},q^{a+b}|q)$.

Moreover, from the above hypergeometric representations we obtain the following values:

$$\begin{aligned}\widetilde{u}_n^{\alpha,\beta}(x(a),a,b)_q =&\frac{(a-b+1|q)_n(2a-\beta+1|q)_n(a-b-\alpha+1|q)_n}{[n]_q!}\\ =&\frac{(q^{a-b+1};q)_n(q^{2a-\beta+1};q)_n(q^{a-b-\alpha+1};q)_n}{q^{\frac{n}{2}(4a-2b-\alpha-\beta+n+1)}\varkappa_q^{2n}(q;q)_n},\end{aligned} \tag{55}$$

$$\begin{aligned}\widetilde{u}_n^{\alpha,\beta}(x(b-1),a,b)_q =&\frac{(a-b+1|q)_n(-2b-\alpha+1|q)_n(a-b-\beta+1|q)_n}{[n]_q!}\\ =&\frac{(q^{a-b+1};q)_n(q^{-2b-\alpha+1};q)_n(q^{-\beta+a-b+1};q)_n}{q^{\frac{n}{2}(2a-4b-\alpha-\beta+n+1)}\varkappa_q^{2n}(q;q)_n}.\end{aligned} \tag{56}$$

In view of (10), we obtain the explicit formula[3]

$$\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q = \frac{\widetilde{\Gamma}_q(s-a+1)\widetilde{\Gamma}_q(s+b+1)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(s+\alpha+b+1)}{\widetilde{\Gamma}_q(s+a+1)\widetilde{\Gamma}_q(s+a-\beta+1)}$$
$$\times\widetilde{\Gamma}_q(b+\alpha-s)\sum_{k=0}^{n}\frac{(-1)^{k+n}[2s+2k-n+1]_q\widetilde{\Gamma}_q(s+k+a+1)\widetilde{\Gamma}_q(2s+k-n+1)}{\widetilde{\Gamma}_q(k+1)\widetilde{\Gamma}_q(n-k+1)\widetilde{\Gamma}_q(2s+k+2)\widetilde{\Gamma}_q(s-n+k-a+1)\widetilde{\Gamma}_q(b-s-k)} \tag{57}$$
$$\times\frac{\widetilde{\Gamma}_q(s+k+a-\beta+1)}{\widetilde{\Gamma}_q(s+k-n+\alpha+b+1)\widetilde{\Gamma}_q(b+\alpha-s-k)\widetilde{\Gamma}_q(s-n+k+b+1)\widetilde{\Gamma}_q(s-n+k-a+\beta+1)}.$$

From this expression follows that

$$\begin{aligned}\widetilde{u}_n^{\alpha,\beta}(x(a),a,b)_q &= \frac{\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(2a-\beta+n+1)\widetilde{\Gamma}_q(b-a+\alpha)}{[n]!\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(2a-\beta+1)\widetilde{\Gamma}_q(b-a+\alpha-n)},\\ \widetilde{u}_n^{\alpha,\beta}(x(b-1),a,b)_q &= \frac{(-1)^n\widetilde{\Gamma}_q(b-a)\widetilde{\Gamma}_q(2b+\alpha)\widetilde{\Gamma}_q(b-a+\beta)}{[n]!\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(2b+\alpha-n)\widetilde{\Gamma}_q(b-a+\beta-n)},\end{aligned} \tag{58}$$

which are in agreement with values (55) and (56) obtained before.

From the hypergeometric representation follows the symmetry property

$$\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q = \widetilde{u}_n^{-b-a+\beta,b+a+\alpha}(x(s),a,b)_q.$$

2.2.2. Differentiation Formulas

Next we use the differentiation formulas (21) and (23) to obtain

$$\frac{\Delta\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q}{\Delta x(s)} = -[2b-2a+\alpha+\beta-n-1]_q\widetilde{u}_{n-1}^{\alpha,\beta}(x(s+\tfrac{1}{2}),a+\tfrac{1}{2},b-\tfrac{1}{2})_q, \tag{59}$$

$$\begin{aligned}[n]_q[2s+1]_q\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q =&\sigma(-s-1)\widetilde{u}_{n-1}^{\alpha,\beta}(x(s+\tfrac{1}{2}),a+\tfrac{1}{2},b-\tfrac{1}{2})_q\\ &-\sigma(s)\widetilde{u}_{n-1}^{\alpha,\beta}(x(s-\tfrac{1}{2}),a+\tfrac{1}{2},b-\tfrac{1}{2})_q,\end{aligned} \tag{60}$$

respectively.

Finally, formulas (24) [or (25)] and (26) lead to the following differentiation formulas:

$$\sigma(s)\frac{\nabla\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q}{[2s]_q} = -\frac{[2b-2a+\alpha+\beta-n-1]_q}{[2b-2a+\alpha+\beta-2n-2]_q}\left[\tau_n(s)\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q-[n+1]_q\widetilde{u}_{n+1}^{\alpha,\beta}(x(s),a,b)_q\right], \tag{61}$$

$$\begin{aligned}\sigma(-s-1)\frac{\Delta\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q}{[2s+2]_q} &= -\frac{[2b-2a+\alpha+\beta-n-1]_q}{[2b-2a+\alpha+\beta-2n-2]_q}\\ \times\Big[(\tau_n(s)+[n]_q[2b-2a+\alpha+\beta-2n-2]_q[2s+1]_q)&\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q-[n+1]_q\widetilde{u}_{n+1}^{\alpha,\beta}(x(s),a,b)_q\Big],\end{aligned} \tag{62}$$

respectively, where $\tau_n(s)$ is given in Table 2.

[3]Obviously formulas (51)–(54) also give two equivalent explicit formulas.

2.3. Dual Set to $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$

To obtain the dual set to $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$, we use the same method as in the previous section.

We start from the orthogonality relation (13) for the polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ defined by (54) and write the dual relation

$$\sum_{n=0}^{N-1} \widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q \widetilde{u}_n^{\alpha,\beta}(x(s'),a,b)_q \frac{1}{d_n^2} = \frac{1}{\rho(s)\Delta x(s-1/2)}\delta_{s,s'}, \quad N=b-a, \tag{63}$$

where ρ and d_n^2 are the weight function and the norm of $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ is given in Table 2. Furthermore, from (54) follows that the functions $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ are polynomials of degree $k=b-s-1$ on the lattice $x(t)=[t]_q[t+1]_q$, where $t=b-a-n+\frac{\alpha+\beta}{2}-1$ (the proof is similar to the one presented in Sec. 2.1.4. and we will omit it here).

To identify the dual set, let us define a new set

$$\begin{aligned}\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q = &\frac{(-1)^k\widetilde{\Gamma}_q(b'-a')\widetilde{\Gamma}_q(b'-a'+\beta')\widetilde{\Gamma}_q(2b'+\alpha')}{[k]!\widetilde{\Gamma}_q(b'-a'-k)\widetilde{\Gamma}_q(b'-a'+\beta'-k)\widetilde{\Gamma}_q(2b'+\alpha'-k)} \\ &\times {}_4\mathrm{F}_3\left(\begin{array}{c} -k, 2a'-2b'-\alpha'-\beta'+k+1, -b'-t, -b'+t+1 \\ a'-b'+1, -2b'-\alpha'+1, a'-b'-\beta'+1 \end{array}\middle|\, q\,,\,1\right),\end{aligned} \tag{64}$$

where

$$k=b-s-1, \quad t=b-a-n+\frac{\alpha+\beta}{2}-1, \quad a'=\frac{\alpha+\beta}{2}, \quad b'=b-a+\frac{\alpha+\beta}{2}, \quad \alpha'=2a-\beta, \quad \beta'=\beta. \tag{65}$$

Obviously they satisfy the following orthogonality relation:

$$\sum_{t=a'}^{b'-1} \widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q \widetilde{\mathrm{u}}_m^{\alpha',\beta'}(x(t),a',b')_q \rho'(t)\Delta x(t-1/2) = (d_k')^2\delta_{k,m}, \tag{66}$$

where now $\rho'(t)$ and d_k' are the weight function ρ and the norm d_n, respectively, given in Table 2 with the corresponding change of the parameters a,b,α,β,n,s by a',b',α',β',k,t (65).

Furthermore, with the above definition (65) for the parameters of $\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q$, the hypergeometric function ${}_4\mathrm{F}_3$ in (64) coincides with the function ${}_4\mathrm{F}_3$ in (54) and, therefore, the following relation between the polynomials $\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')$ and $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ holds:

$$\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q = \widetilde{\mathcal{A}}(\alpha,\beta,a,b,n,s)\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q, \tag{67}$$

where

$$\widetilde{\mathcal{A}}(\alpha,\beta,a,b,n,s) = \frac{(-1)^{b-s-1-n}\widetilde{\Gamma}_q(b-a-n)\widetilde{\Gamma}_q(2b+\alpha-n)\widetilde{\Gamma}_q(b-a+\beta-n)\widetilde{\Gamma}_q(n+1)}{\widetilde{\Gamma}_q(b-s)\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(s+b+\alpha+1)\widetilde{\Gamma}_q(s-a+1)}.$$

To prove that the polynomials $\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q$ are the dual set to $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$, it is sufficient to substitute (67) in (66) and make the change (65) that converts (66) into (46).

Let us also mention that, as in the case of the q-Racah polynomials, TTRR (16) of the polynomials $\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q$ is SODE (4) of the polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$, whereas SODE (4) of $\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q$ converts into TTRR (16) of $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$, and vice versa.

To conclude this section, let us point out that there exists a simple relation connecting both polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ and $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ [see (87) from below]. We will establish it at the end of the next section.

3. Connection with $6j$-Symbols of q-Algebra $SU_q(2)$

3.1. $6j$-Symbols of Quantum Algebra $SU_q(2)$

It is known (see, e.g., [21] and references therein) that the Racah coefficients $U_q(j_1\,j_2\,j\,j_3;j_{12}\,j_{23})$ are used for the transition from the coupling scheme of three angular momenta j_1,j_2,j_3

$$|j_1 j_2(j_{12}),j_3:jm\rangle=\sum_{m_1,m_2,m_3,m_{12}}\langle j_1m_1j_2m_2|j_{12}m_{12}\rangle\langle j_{12}m_{12}j_3m_3|jm\rangle|j_1m_1\rangle|j_2m_2\rangle|j_3m_3\rangle$$

to the following ones:

$$|j_1j_2j_3(j_{23}):jm\rangle=\sum_{m_1,m_2,m_3,m_{23}}\langle j_2m_2j_3m_3|j_{23}m_{23}\rangle\langle j_1m_1j_{23}m_{23}|jm\rangle|j_1m_1\rangle|j_2m_2\rangle|j_3m_3\rangle,$$

where $\langle j_am_aj_bm_b|j_{ab}m_{ab}\rangle$ denotes the Clebsch–Gordan coefficients of the quantum algebra $su_q(2)$. In fact, we have that recoupling is given by

$$|j_1j_2(j_{12}),j_3:jm\rangle=\sum_{j_{23}}U_q(j_1\,j_2\,j\,j_3;j_{12}\,j_{23})|j_1j_2j_3(j_{23}):jm\rangle.$$

The Racah coefficients U define an unitary matrix, i.e., they satisfy the orthogonality relations

$$\sum_{j_{23}}U_q(j_1\,j_2\,j\,j_3;j_{12}\,j_{23})U_q(j_1\,j_2\,j\,j_3;j'_{12}\,j_{23})=\delta_{j_{12},j'_{12}},\tag{68}$$

$$\sum_{j_{12}}U_q(j_1\,j_2\,j\,j_3;j_{12}\,j_{23})U_q(j_1\,j_2\,j\,j_3;j_{12}\,j'_{23})=\delta_{j_{23},j'_{23}}.\tag{69}$$

Usually, instead of the Racah coefficients, it is more convenient to use the $6j$-symbols defined by

$$U_q(j_1\,j_2\,j\,j_3;j_{12}\,j_{23})=(-1)^{j_1+j_2+j_3+j}\sqrt{[2j_{12}+1]_q[2j_{23}+1]_q}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{array}\right\}_q.$$

The $6j$-symbols have the following symmetry property:

$$\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{array}\right\}_q=\left\{\begin{array}{ccc} j_3 & j_2 & j_{23}\\ j_1 & j & j_{12}\end{array}\right\}_q.\tag{70}$$

Here without loss of generality, we suppose that $j_1\ge j_2$ and $j_3\ge j_2$; then for the momenta j_{23} and j_{12} we have the intervals

$$j_3-j_2\le j_{23}\le j_2+j_3,\quad j_1-j_2\le j_{12}\le j_1+j_2,$$

respectively. Now, in order to avoid any other restrictions on these two momenta (caused by the so-called triangle inequalities for the $6j$-symbols), we assume that the following restrictions hold:

$$|j - j_3| \leq \min(j_{12}) = j_1 - j_2, \quad |j - j_1| \leq \min(j_{23}) = j_3 - j_2.$$

3.2. $6j$-Symbols and q-Racah Polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$

Now we are ready to establish the connection of the $6j$-symbols with the q-Racah polynomials.

We fix the variable s as $s = j_{23}$ that runs on the interval $a \leq s \leq b-1$, where $a = j_3-j_2$, $b = j_2+j_3+1$. Let us put

$$(-1)^{j_1+j_{23}+j}\sqrt{[2j_{12}+1]_q}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q = \sqrt{\frac{\rho(s)}{d_n^2}}\, u_n^{\alpha,\beta}(x(s), a, b)_q, \tag{71}$$

where $\rho(s)$ and d_n are the weight function and the norm, respectively, of the q-Racah polynomials on the lattice (1) $u_n^{\alpha,\beta}(x(s), a, b)_q$, and

$$n = j_{12} - j_1 + j_2, \qquad \alpha = j_1 - j_2 - j_3 + j \geq 0, \qquad \beta = j_1 - j_2 + j_3 - j \geq 0.\text{[4]}$$

To verify the above relation, we use the recurrence relation (see Eq. (5.17) in [23])

$$\begin{aligned} &[2]_q[2j_{23}+2]_q A_q^- \left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}-1 \end{array}\right\}_q \\ &\quad - \Big(([2j_{23}]_q[2j_1+2]_q - [2]_q[j-j_{23}+j_1+1]_q[j+j_{23}-j_1]_q) \\ &\quad \times ([2j_2]_q[2j_{23}+2]_q - [2]_q[j_3-j_2+j_{23}+1]_q[j_3+j_2-j_{23}]_q) \\ &\quad - ([2j_2]_q[2j_1+2]_q - [2]_q[j_{12}-j_2+j_1+1]_q[j_{12}+j_2-j_1]_q)[2j_{23}+2]_q[2j_{23}]_q\Big) \\ &\times [2j_{23}+1]_q \left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q + [2]_q[2j_{23}]_q A_q^+ \left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}+1 \end{array}\right\}_q = 0, \end{aligned} \tag{72}$$

where

$$\begin{aligned} A_q^- &= \sqrt{[j+j_{23}+j_1+1]_q[j+j_{23}-j_1]_q[j-j_{23}+j_1+1]_q[j_{23}-j+j_1]_q} \\ &\quad \times \sqrt{[j_2+j_3+j_{23}+1]_q[j_2+j_3-j_{23}+1]_q[j_3-j_2+j_{23}]_q[j_2-j_3+j_{23}]_q}, \\ A_q^+ &= \sqrt{[j+j_{23}+j_1+2]_q[j+j_{23}-j_1+1]_q[j-j_{23}+j_1]_q[j_{23}-j+j_1+1]_q} \\ &\quad \times \sqrt{[j_2+j_3+j_{23}+2]_q[j_2+j_3-j_{23}]_q[j_3-j_2+j_{23}+1]_q[j_2-j_3+j_{23}+1]_q}. \end{aligned} \tag{73}$$

Notice that

$$A_q^- = \sqrt{\sigma(j_{23})\sigma(-j_{23})}, \quad A_q^+ = \sqrt{\sigma(j_{23}+1)\sigma(-j_{23}-1)},$$

[4]Notice that this is equivalent to the following setting:

$$j_1 = (b-a-1+\alpha+\beta)/2, \quad j_2 = (b-a-1)/2, \quad j_3 = (a+b-1)/2,$$
$$j_{12} = (2n+\alpha+\beta)/2, \quad j_{23} = s, \quad j = (a+b-1+\alpha-\beta)/2.$$

where

$$\sigma(j_{23}) = [j_{23} - j_3 + j_2]_q[j_{23} + j_2 + j_3 + 1]_q[j_{23} - j_1 + j]_q[j + j_1 - j_{23} + 1]_q,$$
$$\sigma(-j_{23} - 1) = [j_{23} + j_3 - j_2 + 1]_q[j_2 + j_3 - j_{23}]_q[j_{23} + j_1 - j + 1]_q[j + j_1 + j_{23} + 2]_q.$$

Substituting (71) in (72) and simplifying the expression obtained we arrive at

$$[2s]_q\sigma(-s-1)u_n^{\alpha,\beta}(x(s+1),a,b)_q + [2s+2]_q\sigma(s)u_n^{\alpha,\beta}(x(s-1),a,b)_q$$
$$+ \Big(\lambda_n[2s]_q[2s+1]_q[2s+2]_q - [2s]_q\sigma(-s-1) - [2s+2]_q\sigma(s)\Big)u_n^{\alpha,\beta}(x(s),a,b)_q = 0,$$

which is the difference equation for the q-Racah polynomials (4).

Since $u_0^{\alpha,\beta}(x(s),a,b)_q = 1$, relation (71) leads to

$$(-1)^{j_1+j_{23}+j}\sqrt{[2j_1-2j_2+1]_q}\left\{\begin{matrix} j_1 & j_2 & j_1-j_2 \\ j_3 & j & j_{23}\end{matrix}\right\}_q = \sqrt{\frac{\rho(s)}{d_0^2}} \quad \Rightarrow$$

$$\left\{\begin{matrix} j_1 & j_2 & j_1-j_2 \\ j_3 & j & j_{23}\end{matrix}\right\}_q := \left\{\begin{matrix} j_1 & j_2 & j_1-j_2 \\ j_3 & j & s\end{matrix}\right\}_q$$
$$=(-1)^{j+j_1+s}\sqrt{\frac{[j_1+j+s+1]_q![j_1+j-s]_q![j_1-j+s]_q![j_3-j_2+s]_q!}{[j-j_1+s]_q![j_3+j_2-s]_q![j_2-j_3+s]_q![j_2+j_3+s+1]_q!}}$$
$$\times\sqrt{\frac{[2j_1-2j_2]_q![2j_2]_q![j_2+j_3+j-j_1]_q!}{[2j_1+1]_q![j_1+j_3-j_2-j]_q![j_1-j_3-j_2+j]_q![j_1+j_3-j_2+j+1]_q!}}.$$

Furthermore, substituting the values $s = a$ and $s = b - 1$ in (71) and using (41) we find

$$\left\{\begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_3-j_2\end{matrix}\right\}_q = (-1)^{j_{12}+j_3+j}$$
$$\times\sqrt{\frac{[j_{12}+j_3-j]_q![2j_2]_q![j_{12}+j_3+j+1]_q![2j_3-2j_2]_q![j_2-j_1+j_{12}]_q!}{[j_1-j_2+j_3-j]_q![j_1+j_2-j_{12}]_q![j_1-j_2+j_3+j+1]_q!}}$$
$$\times\sqrt{\frac{[j_1+j_2-j_3+j]_q![j_1-j_2+j_{12}]_q![j_3-j_{12}+j]_q!}{[2j_3+1]_q![j_3-j_1-j_2+j]_q![j_{12}-j_3+j]_q![j_1+j_2+j_{12}+1]_q!}} \tag{74}$$

and

$$\left\{\begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_2+j_3\end{matrix}\right\}_q = (-1)^{j_1+j_2+j_3+j}$$
$$\times\sqrt{\frac{[2j_2]_q![j_{12}-j_3+j]_q![j_2-j_1+j_3+j]_q![2j_3]_q![j_1+j_2+j_3-j]_q!}{[j_1+j_2-j_{12}]_q![j_1-j_2-j_3+j]_q![j_3-j_{12}+j]_q!}}$$
$$\times\sqrt{\frac{[j_2-j_1+j_{12}]_q![j_1-j_2+j_{12}]_q![j_1+j_2+j_3+j+1]_q!}{[2j_2+2j_3+1]_q![j_{12}+j_3-j]_q![j_1+j_2+j_{12}+1]_q![j_{12}+j_3+j+1]_q!}}, \tag{75}$$

which are in agreement with the results of [21].

Relation (71) allows us to obtain several recurrence relations for the $6j$-symbols of the quantum algebra $SU_q(2)$ by using the properties of the q-Racah polynomials. So, TTRR (16) gives

$$\begin{aligned}&[2j_{12}]_q\,\widetilde{A}_q^{+}\left\{\begin{matrix} j_1 & j_2 & j_{12}+1\\ j_3 & j & j_{23}\end{matrix}\right\}_q+[2j_{12}+2]_q\,\widetilde{A}_q^{-}\left\{\begin{matrix} j_1 & j_2 & j_{12}-1\\ j_3 & j & j_{23}\end{matrix}\right\}_q\\ &-\Big([2j_{12}]_q[2j_{12}+1]_q[2j_{12}+2]_q\big([j_{23}]_q[j_{23}+1]_q-[j_3-j_2]_q[j_3-j_2+1]_q\big)+[2j_{12}]_q\\ &\times[j_1-j_2+j_{12}+1]_q[j_{12}-j_1-j_2]_q[j_{12}+j_3-j+1]_q[j_{12}+j_3+j+2]_q-[2j_{12}+2]_q\\ &\times[j_{12}-j_3+j]_q[j_1+j_2+j_{12}+1]_q[j_3-j_{12}+j+1]_q[j_2-j_1+j_{12}]_q\Big)\left\{\begin{matrix} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{matrix}\right\}_q=0,\end{aligned}\tag{76}$$

where

$$\begin{aligned}\widetilde{A}_q^{-}=&\sqrt{[j_2-j_1+j_{12}]_q[j_1-j_2+j_{12}]_q[j_{12}-j_3+j]_q[j_{12}+j_3-j]_q[j_1+j_2+j_{12}+1]_q}\\ &\times\sqrt{[j_{12}+j_3+j+1]_q[j_1+j_2-j_{12}+1]_q[j_3-j_{12}+j+1]_q},\\ \widetilde{A}_q^{+}=&\sqrt{[j_2-j_1+j_{12}+1]_q[j_1-j_2+j_{12}+1]_q[j_{12}-j_3+j+1]_q[j_{12}+j_3-j+1]_q}\\ &\times\sqrt{[j_1+j_2+j_{12}+2]_q[j_{12}+j_3+j+2]_q[j_1+j_2-j_{12}]_q[j_3-j_{12}+j]_q}.\end{aligned}\tag{77}$$

Expressions (42) and (43) yield

$$\begin{aligned}&\sqrt{\sigma(j_{23}+1)}\left\{\begin{matrix} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}+1\end{matrix}\right\}_q+\sqrt{\sigma(-j_{23}-1)}\left\{\begin{matrix} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{matrix}\right\}_q\\ &=[2j_{23}+2]_q\sqrt{[j_2-j_1+j_{12}]_q[j_1-j_2+j_{12}+1]_q}\left\{\begin{matrix} j_1+\frac12 & j_2-\frac12 & j_{12}\\ j_3 & j & j_{23}+\frac12\end{matrix}\right\}_q\end{aligned}\tag{78}$$

and

$$\begin{aligned}&\sqrt{\sigma(-j_{23}-1)}\left\{\begin{matrix} j_1+\frac12 & j_2-\frac12 & j_{12}\\ j_3 & j & j_{23}+\frac12\end{matrix}\right\}_q+\sqrt{\sigma(j_{23})}\left\{\begin{matrix} j_1+\frac12 & j_2-\frac12 & j_{12}\\ j_3 & j & j_{23}-\frac12\end{matrix}\right\}_q\\ &\qquad=[2j_{23}+1]_q\sqrt{[j_{12}-j_1+j_2]_q[j_{12}+j_1-j_2+1]_q}\left\{\begin{matrix} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{matrix}\right\}_q,\end{aligned}\tag{79}$$

respectively, whereas the differentiation formulas (44)–(45) give

$$\begin{aligned}&[2j_{12}+2]_qA_q^{-}\left\{\begin{matrix} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}-1\end{matrix}\right\}_q+[2j_{23}]_q\widetilde{A}_q^{+}\left\{\begin{matrix} j_1 & j_2 & j_{12}+1\\ j_3 & j & j_{23}\end{matrix}\right\}_q\\ &+\Big(\sigma(j_{23})[2j_{12}+2]_q+[j_1-j_2+j_{12}+1]_q[2j_{23}]_q\Lambda(j_{12},j_{23},j_1,j_2)\left\{\begin{matrix} j_1 & j_2 & j_{12}\\ j_3 & j & j_{23}\end{matrix}\right\}_q=0\end{aligned}\tag{80}$$

and

$$
\begin{aligned}
&[2j_{12}+2]_q A_q^{+}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}+1 \end{array}\right\}_q - [2j_{23}+2]_q \widetilde{A}_q^{+}\left\{\begin{array}{ccc} j_1 & j_2 & j_{12}+1 \\ j_3 & j & j_{23} \end{array}\right\}_q \\
&+\Big([2j_{12}+2]_q\sigma(-j_{23}-1) - [2j_{23}+2]_q[j_1-j_2+j_{12}+1]_q\,(\Lambda(j_{12},j_{23},j_1,j_2) \\
&\quad + [j_{12}-j_1+j_2]_q[2j_{12}+2]_q[2j_{23}+1]_q)\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q = 0,
\end{aligned} \tag{81}
$$

respectively, where $A_q^{\pm}$ are given by (73), $\widetilde{A}_q^{\pm}$ by (77), and

$$
\begin{aligned}
\Lambda(j_{12},j_{23},j_1,j_2) = &\sigma\left(\frac{-j_{12}+j_1-j_2}{2}-1\right) - \sigma\left(\frac{-j_{12}+j_1-j_2}{2}\right) \\
&- [2j_{12}+2]_q\left[j_{23}+\frac{j_{12}-j_1+j_2}{2}\right]_q\left[j_{23}+\frac{j_{12}-j_1+j_2}{2}+1\right]_q .
\end{aligned}
$$

Using the hypergeometric representations (35) and (37) we obtain the representation of the $6j$-symbols in terms of the q-hypergeometric function[5] (28)

$$
\begin{aligned}
\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q &= (-1)^{j_{12}+j_{23}+j_2+j}\frac{[2j_2]_q!}{[j_1-j_2+j_3-j]_q![j_1-j_2+j_3+j+1]_q!} \\
&\times\sqrt{\frac{[j_1+j+j_{23}+1]_q![j_1+j-j_{23}]_q![j_1-j+j_{23}]_q![j_3-j_2+j_{23}]_q!}{[j-j_1+j_{23}]_q![j_3+j_2-j_{23}]_q![j_2-j_3+j_{23}]_q![j_2+j_3+j_{23}+1]_q!}} \\
&\times\sqrt{\frac{[j_{12}-j_1+j_2]_q![j_{12}+j_1-j_2]_q![j_3+j-j_{12}]_q![j_3+j_{12}-j]_q![j_3+j_{12}+j+1]_q!}{[j_{12}-j_3+j]_q![j_1+j_2+j_{12}+1]_q![j_1+j_2-j_{12}]_q!}} \\
&\times {}_4\mathrm{F}_3\left(\left.\begin{array}{c} j_1-j_2-j_{12},\, j_1-j_2+j_{12}+1,\, j_3-j_2-j_{23},\, j_{23}+j_3-j_2+1 \\ -2j_2,\ j_1-j_2+j_3-j+1,\ j_1-j_2+j_3+j+2 \end{array}\right| q\,,\,1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q &= (-1)^{j_1+j_{23}+j}\frac{[2j_2]_q![j_2+j_3-j_1+j]_q!}{[j_1-j_2-j_3+j]_q!} \\
&\times\sqrt{\frac{[j_1+j+j_{23}+1]_q![j_1+j-j_{23}]_q![j_1-j+j_{23}]_q![j_3-j_2+j_{23}]_q!}{[j-j_1+j_{23}]_q![j_3+j_2-j_{23}]_q![j_2-j_3+j_{23}]_q![j_2+j_3+j_{23}+1]_q!}} \\
&\times\sqrt{\frac{[j_{12}-j_1+j_2]_q![j_{12}+j_1-j_2]_q![j_{12}-j_3+j]_q!}{[j_1+j_2+j_{12}+1]_q![j_1+j_2-j_{12}]_q![j_3+j-j_{12}]_q![j_{12}+j_3-j]_q![j_3+j_{12}+j+1]_q!}} \\
&\times {}_4\mathrm{F}_3\left(\left.\begin{array}{c} j_1-j_2-j_{12},\, j_1-j_2+j_{12}+1,\, -j_3-j_2+j_{23},\, -j_{23}-j_3-j_2-1 \\ -2j_2,\ j_1-j_2-j_3+j+1,\ j_1-j_2-j_3-j \end{array}\right| q\,,\,1\right).
\end{aligned}
$$

[5]To obtain the representation in terms of the basic hypergeometric series, it is sufficient to use relation (29).

Notice that values (74) and (75) immediately follow from the above representations.

Notice also that the above formulas give two alternative explicit formulas for computing the $6j$-symbols.

The third explicit formula follows from (40):

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = \sqrt{\frac{[j_{23}+j_2-j_3]_q![j_{23}+j_2+j_3+1]_q![j_{23}+j-j_1]_q![j_2+j_3-j_{23}]_q!}{[j_{23}+j_3-j_2]_q![j_{23}+j_1-j]_q![j_{23}+j_1+j+1]_q![j_1+j-j_{23}]_q!}}$$
$$\times \sqrt{\frac{[j_{12}-j_1+j_2]_q![j_1-j_2+j_{12}]_q![j_1+j_2-j_{12}]_q![j_3-j_{12}+j]_q!}{[j_{12}-j_3+j]_q![j_{12}+j_3-j]_q![j_1+j_2+j_{12}+1]_q![j_{12}+j_3+j]_q!}}$$
$$\times \sum_{k=0}^{j_{12}-j_1+j_2} \frac{(-1)^{k+j_1+j_{23}+j}[2k+j_1-j_2-j_{12}+2j_{23}+1]_q[k+j_{23}+j_3-j_2]_q!}{[k]_q![j_{12}-j_1+j_2-k]_q![2j_3+1+k]_q![k+j_{23}+j_1-j_{12}-j_3]_q!}$$
$$\times \frac{[2j_{23}+k-j_{12}+j_1-j_2]_q![k+j_{23}+j_1-j]_q![k+j_{23}+j_1+j+1]_q![j_1+j-j_{23}-k]_q!}{[k+j_{23}+j_1-j_{12}+j_3+1]_q![k+j_{23}+j-j_2-j_{12}]_q![j_2+j_3-j_{23}+1-k]_q!}.$$

To conclude this section, let us point out that the orthogonality relations (68) and (69) lead to the orthogonality relations for the Racah polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$ (35) and their duals $\mathrm{u}_k^{\alpha',\beta'}(x(t),a',b')_q$, respectively, and also that relation (50) between the q-Racah and dual q-Racah polynomials corresponds to the symmetry property (70).

3.3. $6j$-Symbols and Alternative q-Racah Polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$

In this section, we provide the same comparative analysis but for the alternative q-Racah polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$.

We again choose $s=j_{23}$ that runs on the interval $[a,b-1]$, $a=j_3-j_2$, $b=j_2+j_3+1$.

In this case, the connection is given by the formula

$$(-1)^{j_{12}+j_3+j}\sqrt{[2j_{12}+1]_q} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = \sqrt{\frac{\rho(s)}{d_n^2}}\, \widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q, \tag{82}$$

where $\rho(s)$ and d_n are the weight function and the norm, respectively, of the alternative q-Racah polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ (see Sec. 2.2.) on the lattice (1) and

$$n=j_1+j_2-j_{12}, \qquad \alpha=j_1-j_2-j_3+j\ge 0, \qquad \beta=j_1-j_2+j_3-j\ge 0.$$

In view of the above relations, one sees that SODE (4) for the polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ converts into the recurrence relation (72), as well as TTRR (16) converts into the recurrence relation (76). Evaluating (82) in $s=j_{23}=j_3-j_2$ and $s=j_{23}=j_2+j_3+1$ and using (58) we recover values (74) and (75), respectively. If we now put $n = 0$, i.e., $j_{12} = j_1 + j_2$, we obtain the value

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_1+j_2 \\ j_3 & j & j_{23} \end{array} \right\}_q := \left\{ \begin{array}{ccc} j_1 & j_2 & j_1+j_2 \\ j_3 & j & s \end{array} \right\}_q$$

$$= (-1)^{j_1+j_2+j_3+j} \sqrt{\frac{[2j_1]_q![2j_2]_q![j_1+j_2+j_3+j+1]_q![j_1+j_2-j_3+j]_q!}{[2j_1+2j_2+1]_q![-j_1-j_2+j_3+j]_q![j_2+j_3+s+1]_q!}}$$

$$\times \sqrt{\frac{[s-j_1+j]_q![s-j_2+j_3]_q!}{[j_1+j-s]_q![j_1-j+s]_q![j_1+j+s+1]_q![j_2+j_3-s]_q![j_2-j_3+s]_q!}}.$$

Expressions (59) and (60) yield

$$\sqrt{\varsigma(j_{23}+1)} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}+1 \end{array} \right\}_q - \sqrt{\varsigma(-j_{23}-1)} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q$$
$$= [2j_{23}+2]_q \sqrt{[j_1+j_2-j_{12}]_q[j_1+j_2+j_{12}+1]_q} \left\{ \begin{array}{ccc} j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_{12} \\ j_3 & j & j_{23}+\frac{1}{2} \end{array} \right\}_q \tag{83}$$

and

$$\sqrt{\varsigma(-j_{23}-1)} \left\{ \begin{array}{ccc} j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_{12} \\ j_3 & j & j_{23}+\frac{1}{2} \end{array} \right\}_q - \sqrt{\varsigma(j_{23})} \left\{ \begin{array}{ccc} j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_{12} \\ j_3 & j & j_{23}-\frac{1}{2} \end{array} \right\}_q$$
$$= [2j_{23}+1]_q \sqrt{[j_1+j_2-j_{12}]_q[j_1+j_2+j_{12}+1]_q} \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q, \tag{84}$$

respectively, where

$$\varsigma(j_{23}) = [j_{23}-j_3+j_2]_q[j_{23}+j_2+j_3+1]_q[j_{23}-j_1+j+1]_q[j+j_1+j_{23}+1]_q,$$
$$\varsigma(-j_{23}-1) = [j_{23}+j_3-j_2+1]_q[j_2+j_3-j_{23}]_q[j_{23}+j_1-j+1]_q[j+j_1-j_{23}]_q.$$

Differentiation formulas (61)–(62) give

$$[2j_{12}]_q A_q^- \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}-1 \end{array} \right\}_q - [2j_{23}]_q \widetilde{A}_q^- \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12}-1 \\ j_3 & j & j_{23} \end{array} \right\}_q$$
$$-\Big(\varsigma(j_{23})[2j_{12}]_q + [j_1+j_2+j_{12}+1]_q[2j_{23}]_q\widetilde{\Lambda}(j_{12},j_{23},j_1,j_2)\Big) \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = 0 \tag{85}$$

and

$$[2j_{12}]_q A_q^+ \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23}+1 \end{array} \right\}_q + [2j_{23}+2]_q \widetilde{A}_q^- \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12}-1 \\ j_3 & j & j_{23} \end{array} \right\}_q$$
$$-\Big([2j_{12}]_q\varsigma(-j_{23}-1) - [2j_{23}+2]_q[j_1+j_2+j_{12}+1]_q\Big(\widetilde{\Lambda}(j_{12},j_{23},j_1,j_2) \tag{86}$$
$$+[j_1+j_2-j_{12}]_q[2j_{12}]_q[2j_{23}+1]_q\Big)\Big) \left\{ \begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array} \right\}_q = 0,$$

respectively, where $A_q^{\pm}$ are given by (73), $\widetilde{A}_q^{\pm}$ by (77), and

$$\widetilde{\Lambda}(j_{12}, j_{23}, j_1, j_2) = \varsigma\left(\frac{j_{12}-j_1-j_2}{2}-1\right) - \varsigma\left(\frac{j_{12}-j_1-j_2}{2}\right) - [2j_{12}]_q \left[j_{23}+\frac{j_1+j_2-j_{12}}{2}\right]_q \left[j_{23}+\frac{j_1+j_2-j_{12}}{2}+1\right]_q .$$

If we now use the hypergeometric representations (52) and (54), we obtain two new representations of the $6j$-symbols in terms of the q-hypergeometric function (28)

$$\left\{\begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix}\right\}_q = (-1)^{j_{12}+j_3+j} \frac{[2j_2]_q![j_1+j_2-j_3+j]_q!}{[j_3-j_2-j_1+j]_q!}$$
$$\times \sqrt{\frac{[j-j_1+j_{23}]_q![j_3-j_2+j_{23}]_q!}{[j_1+j+j_{23}+1]_q![j_1+j-j_{23}]_q![j_2-j_3+j_{23}]_q![j_2+j_3+j_{23}+1]_q![j_1-j+j_{23}]_q!}}$$
$$\times \sqrt{\frac{[j_3-j_{12}+j]_q![j_{12}+j_3-j]_q![j_3+j_{12}+j+1]_q![j_1-j_2+j_{12}+1]_q!}{[j_3+j_2-j_{23}]_q![j_1+j_2-j_{12}]_q![j_2-j_1+j_{12}]_q![j_{12}-j_3+j]_q![j_1+j_2+j_{12}+1]_q!}}$$
$$\times {}_4\mathrm{F}_3\left(\begin{matrix} j_{12}-j_1-j_2, -j_1-j_2-j_{12}-1, j_3-j_2-j_{23}, j_{23}+j_3-j_2+1 \\ -2j_2,\ j_3-j_1-j_2+j+1,\ j_3-j_1-j_2-j \end{matrix}\middle| q\,,1\right)$$

and

$$\left\{\begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix}\right\}_q = (-1)^{j_1+j_{23}+j} \frac{[2j_2]_q![j_1+j_2+j_3+j]_q![j_1+j_2+j_3-j]_q!}{\sqrt{[j_3+j_2-j_{23}]_q![j_1+j_2-j_{12}]_q!}}$$
$$\times \sqrt{\frac{[j-j_1+j_{23}]_q![j_3-j_2+j_{23}]_q!}{[j_1+j+j_{23}+1]_q![j_1+j-j_{23}]_q![j_2-j_3+j_{23}]_q![j_2+j_3+j_{23}+1]_q![j_1-j+j_{23}]_q!}}$$
$$\times \sqrt{\frac{[j_{12}-j_3+j]_q![j_{12}+j_1-j_2+1]_q!}{[j_3-j_{12}+j]_q![j_{12}+j_3-j]_q![j_1+j_2+j_{12}+1]_q![j_2-j_1+j_{12}]_q![j_3+j_{12}+j+1]_q!}}$$
$$\times {}_4\mathrm{F}_3\left(\begin{matrix} j_{12}-j_1-j_2, -j_1-j_2-j_{12}-1, -j_3-j_2-j_{23}-1, j_{23}-j_3-j_2 \\ -2j_2,\ -j_1-j_2-j_3-j,\ j-j_1-j_2-j_3 \end{matrix}\middle| q\,,1\right).$$

Notice that values (74) and (75) also follow from the above representations.

Obviously, the above formulas give two other alternative explicit formulas for computing the $6j$-symbols.

Finally, from (57) follows

$$\left\{\begin{array}{ccc} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{array}\right\}_q = \sqrt{\frac{[j_{23}+j_1+j+1]_q![j_1-j_{23}+j]_q![j_2-j_3+j_{23}]_q![j_2+j_3+j_{23}+1]_q!}{[j_{23}-j_2+j_3]_q![-j_1+j_{23}+j]_q![-j_{12}+j_3+j]_q![j_1+j_2+j_{12}+1]_q!}}$$
$$\times\sqrt{\frac{[j_1+j_{23}-j]_q![j_2+j_3-j_{23}]_q![j_1+j_2-j_{12}]_q![-j_1+j_2+j_{12}]_q!}{[-j_3+j+j_{12}]_q!^{-1}[j_{12}+j_3+j+1]_q!^{-1}[j_1-j_2+j_{12}]_q!^{-1}}}$$
$$\times\sum_{l=0}^{j_1+j_2-j_{12}}\frac{(-1)^{l+j_1+j_2+j_3+j}[2j_{23}+2l-j_1-j_2+j_{12}+1]_q}{[l]_q![j_1+j_2-j_{12}-l]_q![2j_{23}+l+1]_q![j_{23}-j_1-j_2+j_{12}+l+j_2-j_3]_q![j_2+j_3-j_{23}-l]_q!}$$
$$\times\frac{[2j_{23}+l-j_1-j_2+j_{12}]_q![j_{23}+l-j_2+j_3]_q![-j_1+j+j_{23}+l]_q!}{[l+j_{12}-j_2+j+j_{23}+1]_q![j_1+j-j_{23}-l]_q![-j_1+j_{12}+j_3+j_{23}+l+1]_q![-j_2+j_{12}-j+j_{23}+l]_q!}.$$

To conclude this section, let us point out that the orthogonality relations (68) and (69) lead to the orthogonality relations for the alternative Racah polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$ (52) and their duals $\widetilde{\mathrm{u}}_k^{\alpha',\beta'}(x(t),a',b')_q$ (64), respectively, and relation (67) between the q-Racah and dual q-Racah polynomials corresponds to the symmetry property (70).

3.4. Connection between $\widetilde{u}_k^{\alpha,\beta}(x(s),a,b)_q$ and $u_n^{\alpha,\beta}(x(s),a,b)_q$

Let us obtain the formula connecting the two families $\widetilde{u}_k^{\alpha,\beta}(x(s),a,b)_q$ and $u_n^{\alpha,\beta}(x(s),a,b)_q$.

In fact, Eqs. (71) and (82) suggest the following relation between both Racah polynomials $\widetilde{u}_k^{\alpha,\beta}(x(s),a,b)_q$ and $u_n^{\alpha,\beta}(x(s),a,b)_q$:

$$\widetilde{u}_{b-a-1-n}^{\alpha,\beta}(x(s),a,b)_q = (-1)^{s-a-n} \times\frac{\widetilde{\Gamma}_q(s-a+\beta+1)\widetilde{\Gamma}_q(b+\alpha-s)\widetilde{\Gamma}_q(b+\alpha+1+s)\widetilde{\Gamma}_q(a+b-\beta-n)}{\widetilde{\Gamma}_q(s+a-\beta+1)\widetilde{\Gamma}_q(\alpha+1+n)\widetilde{\Gamma}_q(\beta+1+n)\widetilde{\Gamma}_q(a+b+\alpha+1+n)}\,u_n^{\alpha,\beta}(x(s),a,b)_q. \tag{87}$$

To prove this, it is sufficient to substitute the above formula into the difference equation (4) of the polynomials $\widetilde{u}_n^{\alpha,\beta}(x(s),a,b)_q$.

After some straightforward computations, the resulting difference equation converts into the corresponding difference equation for the polynomials $u_n^{\alpha,\beta}(x(s),a,b)_q$.

Notice that from this relation follows that

$${}_4\mathrm{F}_3\left(\begin{array}{c} a-b+n+1, a-b-\alpha-\beta-n, a-s, a+s+1 \\ a-b+1, 2a-\beta+1, a-b-\alpha+1 \end{array}\middle|\, q\,,\,1\right)$$
$$=\frac{(\beta+1|q)_{s-a}(b+\alpha+a+1|q)_{s-a}}{(2a-\beta+1|q)_{s-a}(a-b-\alpha+1)_{s-a}}\,{}_4\mathrm{F}_3\left(\begin{array}{c} -n, \alpha+\beta+n+1, a-s, a+s+1 \\ a-b+1, \beta+1, a+b+\alpha+1 \end{array}\middle|\, q\,,\,1\right).$$

This provides the following identity for terminating the ${}_4\phi_3$ basic series $(n, N-n-1, k=0,1,2,\ldots)$:

$${}_4\varphi_3\left(\begin{array}{c} q^{n-N+1}, q^{-n-N+1}A^{-1}B^{-1}, q^{-k}, q^{-k}D \\ q^{1-N}, q^{-2k}DB^{-1}, q^{1-N}A^{-1} \end{array}\middle|\, q\,,\,q\right)$$
$$=\frac{q^{-kN}}{A^kB^k}\frac{(qB;q)_k(q^{N-2k}DA;q)_k}{(q^{-2k}DB^{-1};q)_k,(q^{1-N}A^{-1};q)_k}\,{}_4\varphi_3\left(\begin{array}{c} q^{-n}, ABq^n, q^{-k}, q^{-k}D \\ q^{1-N}, qB, q^{N-2k}DA \end{array}\middle|\, q\,,\,q\right).$$

4. Conclusions

Here we have provided a detailed study of two kinds of Racah q-polynomials on the lattice $x(s) = [s]_q[s+1]_q$ and also their comparative analysis with the Racah coefficients or $6j$-symbols of the quantum algebra $U_q(su(2))$.

To conclude the paper, we will briefly discuss the relation of the q-Racah polynomials to the representation theory of the quantum algebra $U_q(su(3))$. In [9] (see § 5.5.3 therein) it was shown that the transformation between two different bases (λ, μ) of the irreducible representation of the classical (not q) algebra $su(3)$ corresponding to the reductions $su(3) \supset su(2) \times u(1)$ and $su(3) \supset u(1) \times su(2)$ of the $su(3)$ algebra in two different subalgebras $su(2)$ is given in terms of the Weyl coefficients that are, up to a sign (phase), the Racah coefficients of the algebra $su(2)$. The same statement can be made in the case of the quantum algebra $su_q(3)$ [32, 33]. The Weyl coefficients of the transformation between two bases of the irreducible representation (λ, μ) corresponding to the reductions $su_q(3) \supset su_q(2) \times u_q(1)$ and $su_q(3) \supset u_q(1) \times su_q(2)$ of the quantum algebra $su_q(3)$ in two different quantum subalgebras $su_q(2)$ coincide (up to a sign) with the q-Racah coefficients of $su_q(2)$.

In fact, the Weyl coefficients satisfy certain difference equations that are equivalent to the differentiation formulas for the q-Racah polynomials $u_n^{\alpha,\beta}(x(s), a, b)_q$ and $\widetilde{u}_n^{\alpha,\beta}(x(s), a, b)_q$; so, following the idea of [9] (see § 5.5.3 therein), we can assure that the main properties of the q-Racah polynomials are closely related to the representations of the quantum algebra $U_q(su(3))$. Finally, let us point out that the same assertion can be made but with the noncompact quantum algebra $U_q(su(2, 1))$. This will be carefully done in a forthcoming paper.

The results obtained can be used in models of photon–atom interactions which employ polynomial Hamiltonians. In these models, structures like the q-analogs of Racah polynomials naturally appear in quantum optics.

Acknowledgments

This research has been supported by the DGES grant BFM 2003-06335-C03 (RAN, RCS) and the PAI grant FQM-0262 (RAN). Yu. F. S. is thankful to the Russian Foundation for Basic Research for partial support under Project No. 02-01-00668.

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