

Limit relations between q -Krall type orthogonal polynomials

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Abstract

In this paper, we consider a natural extension of several results related to Krall-type polynomials introducing a modification of a q -classical linear functional via the addition of one or two mass points. The limit relations between the q -Krall type modification of big q -Jacobi, little q -Jacobi, big q -Laguerre, and other families of the q -Hahn tableau are established.

1 Introduction

In the last years, perturbations of a linear functional \mathcal{C} via the addition of Dirac delta functions —the so-called Krall-type orthogonal polynomials— have been intensively studied (for recent reviews see e.g. [6, 7] and references therein), i.e. $\mathcal{U} = \mathcal{C} + A\delta(x - x_0)$, where $A \geq 0$, $x_0 \in \mathbb{R}$ and $\delta(x - y)$ means the Dirac linear functional defined by $\langle \delta(x - y), p(x) \rangle = p(y)$, $\forall p \in \mathbb{C}[x]$, the linear space of polynomials with complex coefficients. Of particular interest are the cases when the starting functional is a classical linear functional (Jacobi [6, 18], Laguerre [6, 16], Hermite [6], and Bessel [20]) and a discrete one (Hahn, Meixner, Kravchuk, and Charlier) [3, 4, 5, 10, 11, 14]. A more general case $\mathcal{U} = \mathcal{C} + \sum_{i=1}^M A_i \delta(x - a_i) - \sum_{j=1}^N B_j \delta'(x - b_j)$ was studied in a recent paper [2] where a special emphasis is given when \mathcal{C} is a semiclassical linear functional.

In a recent paper [9] the case when \mathcal{C} is a discrete semiclassical or q -semiclassical linear functional was considered in details. Here we will focus our attention on the case when \mathcal{C} is a q -classical linear functional and we will construct the Krall-type polynomials associated with the q -classical families of the so-called q -Hahn tableau [8, 19]. This case is not so well known and only few papers deals with examples of such polynomials: the Stieltjes-Wigert polynomials [12], a particular case of the q -little Jacobi polynomials [24], and the Al-Salam & Carlitz I and discrete q -Hermite I [9].

The aim of the present contribution¹ is to continue the work started in [9] and study several families of q -Krall type orthogonal polynomials. In particular, we will obtain the limits of the q -Krall type polynomials in the q -Hahn tableau. In such a way we will continue the study started in [6] concerning the limit relations among the Krall-type families.

¹This paper appear in R. Álvarez-Nodarse and R.S. Costas-Santos, Limit relations between q -Krall type orthogonal polynomials, *J. Math. Anal. Appl.* **322**(1) (2006) 158-176, but we have found a misprint in formulas (5) and (6). In this version we correct the misprints as well as all all formulas that changes as a consequence of such misprint. The changes are in blue.

The structure of the paper is the following. In section 2 some preliminaries and the basic parameters of the families that we will consider later on are given. In particular, we include the explicit values for the kernels of the corresponding q -classical polynomials in terms of the polynomials and their q -derivatives. In section 3 the q -Krall-type orthogonal polynomials are defined and some algebraic properties are deduced for these new families. Finally, in section 4, the limits of the modified polynomials of the examples considered in section 2 are established.

2 Preliminary results

In this section, we state some formulas for q -classical orthogonal polynomials $P_n(x(s))_q \equiv P_n(s)_q$ of the q -Hahn tableau, orthogonal with respect to a q -classical linear functional \mathcal{C}_q [21], i.e.,

$$\langle \mathcal{C}_q, P_n P_m \rangle = d_n^2 \delta_{n,m}, \quad d_n^2 \neq 0, \quad n, m = 0, 1, 2, \dots \quad (1)$$

These functionals usually have the form (see the section 2.1 for more details)

$$\langle \mathcal{C}_q, P \rangle = \begin{cases} \sum_{s=0}^{\infty} P(x(s)) \rho(s) \nabla x_1(s), & \text{little } q\text{-Jacobi, } q\text{-Meixner, Wall, } q\text{-Charlier,} \\ \int_{s_0}^{s_1} P(x) \rho(x) d_q x, & \text{big } q\text{-Jacobi, big } q\text{-Laguerre, Al-Salam-Carlitz I,} \end{cases}$$

etc., where $\int_{s_0}^{s_1} f(t) d_q t$ is the Jackson q -integral (see [13, 17]), ρ is a weight function satisfying the following difference equation of Pearson-type

$$\Delta[\sigma(s)\rho(s)] = \tau(s)\rho(s)\nabla x_1(s) \iff \frac{\rho(s+1)}{\rho(s)} = \frac{\sigma(s) + \tau(s)\nabla x_1(s)}{\sigma(s+1)},$$

the lattice is $x(s) = cq^{\pm s} + c'$, $x_k(s) = x(s + \frac{k}{2})$, and ∇ and Δ are the backward and forward difference operators defined respectively as $\nabla f(s) = f(s) - f(s-1)$, $\Delta f(s) = f(s+1) - f(s)$. Now, consider the sequence of q -classical orthogonal polynomials with respect to the linear functional \mathcal{C}_q (q -COP). They satisfy the second order linear difference equation (SODE) of hypergeometric type [22] $\sigma(s) \frac{\Delta}{\nabla x_1(s)} \frac{\nabla y(s)}{\nabla x(s)} + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda_n y(s) = 0$, where $\sigma(s)$ and $\tau(s)$ are polynomials of degree at most 2 and exactly 1, respectively, and λ_n is a constant. Moreover, these families of q -polynomials satisfy several algebraic relations such as a three-term recurrence relation (TTRR)

$$x(s)P_n(s)_q = \alpha_n P_{n+1}(s)_q + \beta_n P_n(s)_q + \gamma_n P_{n-1}(s)_q, \quad n = 0, 1, 2, \dots, \quad (2)$$

with the initial conditions $P_0(s)_q = 1$, $P_{-1}(s)_q = 0$, the structure relations ($n = 1, 2, 3, \dots$)

$$\begin{aligned} \sigma(s) \frac{\nabla P_n(s)_q}{\nabla x(s)} &= \tilde{\alpha}_n P_{n+1}(s)_q + \tilde{\beta}_n P_n(s)_q + \tilde{\gamma}_n P_{n-1}(s)_q, \\ \phi(s) \frac{\Delta P_n(s)_q}{\Delta x(s)} &= \hat{\alpha}_n P_{n+1}(s)_q + \hat{\beta}_n P_n(s)_q + \hat{\gamma}_n P_{n-1}(s)_q, \end{aligned} \quad (3)$$

where $\phi(s) = \sigma(s) + \tau(s)\nabla x_1(s)$, as well as the Christoffel-Darboux formula for the n -th kernel associated with the family

$$K_n(s_1, s_2) := \sum_{m=0}^n \frac{P_m(s_1)_q P_m(s_2)_q}{d_m^2} = \frac{\alpha_n P_{n+1}(s_1)_q P_n(s_2)_q - P_{n+1}(s_2)_q P_n(s_1)_q}{d_n^2 (x(s_1) - x(s_2))}. \quad (4)$$

In the sequel we will use the notation $\mathbb{K}_n(s_0) := K_n(s_0, s_0)$. From (4) and (3) follows that

Table 1: Parameters of big q -Jacobi and Stieltjes-Wigert polynomials

	Big q -Jacobi	Stieltjes-Wigert
$\sigma(x)$	$q^{-1}(x - aq)(x - cq)$	$q^{-1}x$
$\phi(s)$	$aq(x - 1)(bx - c)$	x^2
λ_n	$-q^{\frac{1}{2}} \frac{(1-abq^{n+1})(1-q^n)}{(1-q)^2 q^n}$	$q^{\frac{1}{2}} \frac{1-q^n}{(1-q)^2}$
$\rho(x)$	$\frac{1}{aq(1-q)} \frac{(aq, bq, cq, abc^{-1}q, a^{-1}x, c^{-1}x; q)_\infty}{(q, abq^2, a^{-1}c, ac^{-1}q, x, bc^{-1}x; q)_\infty}$	$-\frac{1}{\ln q} \frac{1}{(q, -x, -qx^{-1}; q)_\infty}$
d_n^2	$\frac{(1-abq)(q, bq, abc^{-1}q; q)_n}{(1-abq^{2n+1})(aq, abq, cq; q)_n} \left(-acq^{\frac{n+3}{2}}\right)^n$	$\frac{1}{(q; q)_n q^n}$
$P_n(x_0)$	$P_n(aq; a, b, c; q) = \frac{(abc^{-1}q; q)_n}{(cq; q)_n} \left(-cq^{\frac{n+1}{2}}\right)^n$	$S_n(0; q) = \frac{1}{(q; q)_n}$
$P_n(x_1)$	$P_n(cq; a, b, c; q) = \frac{(bq; q)_n}{(aq; q)_n} \left(-aq^{\frac{n+1}{2}}\right)^n$	—
$\hat{\alpha}_n/\alpha_n$	$abq \frac{1-q^n}{1-q}$	$\frac{1-q^n}{1-q}$
$\hat{\gamma}_n/\gamma_n$	$-\frac{1-abq^{n+1}}{(1-q)^n q^n}$	$\frac{1}{1-q}$
$\tilde{\alpha}_n/\alpha_n$	$\frac{1-q^n}{(1-q)q^n}$	0
$\tilde{\gamma}_n/\gamma_n$	$-\frac{1-abq^{n+1}}{1-q}$	$\frac{q^n}{1-q}$

1. If $\sigma(s_0) = 0$, then

$$K_n(s, s_0) = \frac{P_n(s_0)_q}{d_n^2 \left(\frac{\tilde{\gamma}_n}{\gamma_n} - \frac{\tilde{\alpha}_n}{\alpha_n}\right)} \left[\frac{\tilde{\gamma}_n}{\gamma_n} P_n(s)_q - \frac{\sigma(s)}{x(s) - x(s_0)} \frac{\nabla P_n(s)_q}{\nabla x(s)} \right]. \quad (5)$$

2. If $\phi(s_0) = 0$, then

$$K_n(s, s_0) = \frac{P_n(s_0)_q}{d_n^2 \left(\frac{\tilde{\gamma}_n}{\gamma_n} - \frac{\tilde{\alpha}_n}{\alpha_n}\right)} \left[\frac{\hat{\gamma}_n}{\gamma_n} P_n(s)_q - \frac{\phi(s)}{x(s) - x(s_0)} \frac{\Delta P_n(s)_q}{\Delta x(s)} \right]. \quad (6)$$

Remark 2.1 A straightforward calculation shows that $\frac{\tilde{\alpha}_n}{\alpha_n}$ and $\frac{\tilde{\gamma}_n}{\gamma_n}$ are independent of the normalization of $P_n(s)_q$, i.e. if $\hat{P}_n(s) = C_n P_n(s)_q$ then those ratios do not change. Moreover ([1, eq. (6.15)]) $\tilde{\gamma}_n/\gamma_n - \tilde{\alpha}_n/\alpha_n = \hat{\gamma}_n/\gamma_n - \hat{\alpha}_n/\alpha_n$.

2.1 The q -classical polynomials

In this section, we will summarize the main properties of the q -polynomials of the q -Hahn tableau needed in the next sections (for more details see [17]). In all cases we have used (5) and (6) for computing the kernels at the corresponding points. In the sequel we will consider probabilistic measures, i.e. $d_0^2 = 1$. This fact will be useful in order to obtain the right limits for the corresponding q -Krall-type polynomials. Here and through out the paper we will use the standard notation for the basic series. For more details see [13].

The big q -Jacobi polynomials $P_n(x; a, b, c; q)$, introduced by Hahn in 1949, are the most general family of q -polynomials on the q -linear lattice $x := x(s) = q^s$. They constitute a q -COP sequence with respect to the linear functional \mathcal{C}^{BqJ}

$$\langle \mathcal{C}^{BqJ}, P \rangle := \int_{cq}^{aq} P(x) \rho(x) d_q x, \quad (7)$$

where the weight function $\rho(x)$, supported on $[cq, aq]$, $0 < a, b < q^{-1}$, $c < 0$, is given in table 1.

Table 2: Parameters of little q -Jacobi, q -Laguerre and Al-Salam-Carlitz I polynomials

	little q -Jacobi	q -Laguerre	Al-Salam-Carlitz I
$\sigma(x)$	$q^{-1}x(x-1)$	$q^{-1}x$	$(x-1)(x-a)$
$\phi(s)$	$ax(bqx-1)$	$ax(x+1)$	a
λ_n	$-q^{\frac{3}{2}} \frac{(1-abq^{n+1})(1-q^n)}{(1-q)^2 q^n}$	$aq^{\frac{1}{2}} \frac{1-q^n}{(1-q)^2}$	$-q^{\frac{3}{2}} \frac{1-q^n}{(1-q)^2 q^n}$
$\rho(x)$	$\frac{(aq;q)_\infty (bq;q)_s a^s}{(abq^2;q)_\infty (q;q)_s}$	$\frac{a^s (aq, -c, -c^{-1}q; q)_\infty}{(q, -caq, -c^{-1}a^{-1}, -cq^s; q)_\infty}$	$\frac{1}{1-q} \frac{(qx, a^{-1}qx; q)_\infty}{(q, a, a^{-1}q; q)_\infty}$
d_n^2	$\frac{(1-abq)}{(1-abq^{2n+1})} \frac{(q, bq; q)_n}{(aq, abq; q)_n} (aq)^n$	$\frac{(aq; q)_n}{(q; q)_n q^n}$	$(q; q)_n \left(-aq^{\frac{n-1}{2}}\right)^n$
$P_n(x_0)$	$p_n(0; a, b q) = 1$	$L_n^{(\alpha)}(0; q) = \frac{(aq; q)_n}{(q; q)_n}$	$U_n^{(a)}(1; q) = \left(-aq^{\frac{n-1}{2}}\right)^n$
$\tilde{\alpha}_n/\alpha_n$	$\frac{1-q^n}{(1-q)q^n}$	0	$q \frac{(1-q^n)}{(1-q)q^n}$
$\tilde{\gamma}_n/\gamma_n$	$-\frac{1-abq^{n+1}}{1-q}$	$\frac{aq^n}{1-q}$	$-\frac{q}{1-q}$

For these polynomials we also need the following expressions for the kernels

$$\begin{aligned}
 K_n^{BqJ}(x, aq) &= \frac{(aq, abq; q)_n}{(q, bq; q)_n} \left[\frac{(1-abq^{n+1})P_n(x; a, b, c; q) - (x-cq)(1-q^{-1})\mathcal{D}_q P_n(q^{-1}x; a, b, c; q)}{(1-abq)a^n} \right], \\
 K_n^{BqJ}(x, 1) &= \frac{(aq, abq, cq; q)_n}{(q, bq, abc^{-1}q; q)_n} \left[\frac{(q^{-n}-abq)P_n(x; a, b, c; q) - aq(bx-c)(q-1)\mathcal{D}_q P_n(x; a, b, c; q)}{(1-abq)(-acq^{n/2+1/2})^n} \right], \\
 K_n^{BqJ}(x, cq) &= \frac{(abq, cq; q)_n}{(q, abc^{-1}q; q)_n} \left[\frac{(1-abq^{n+1})P_n(x; a, b, c; q) - (x-aq)(1-q^{-1})\mathcal{D}_q P_n(q^{-1}x; a, b, c; q)}{(1-abq)c^n} \right],
 \end{aligned} \tag{8}$$

and

$$\begin{aligned}
 \mathbb{K}_n^{BqJ}(aq) &= \sum_{k=0}^n \frac{(1-abq^{2k+1})(aq, abq, abc^{-1}q; q)_k}{(1-abq)(q, bq, cq; q)_k} \left(-a^{-1}cq^{\frac{k-1}{2}}\right)^k, \\
 \mathbb{K}_n^{BqJ}(cq) &= \sum_{k=0}^n \frac{(1-abq^{2k+1})(bq, abq, cq; q)_k}{(1-abq)(q, aq, abc^{-1}q; q)_k} \left(-ac^{-1}q^{\frac{k-1}{2}}\right)^k, \\
 \mathbb{K}_n^{BqJ}(1) &= \sum_{k=0}^n \frac{(1-abq^{2k+1})(aq, abq, cq; q)_k}{(1-abq)(q, bq, abc^{-1}q; q)_k} \left(-acq^{\frac{k+3}{2}}\right)^{-k}, \\
 K_n^{BqJ}(aq, cq) &= \sum_{k=0}^n \frac{(1-abq^{2k+1})(abq; q)_k \left(-q^{\frac{k-1}{2}}\right)^k}{(1-abq)(q; q)_k} = \frac{(abq^2; q)_n}{(q; q)_n} \left(-q^{\frac{n+1}{2}}\right)^n, \\
 K_n^{BqJ}(aq, 1) &= \sum_{k=0}^n \frac{(1-abq^{2k+1})(aq, abq; q)_k (aq)^{-k}}{(1-abq)(q, bq; q)_k} = \frac{(aq^2, abq^2; q)_n}{(q, bq; q)_n} (aq)^{-n},
 \end{aligned}$$

where \mathcal{D}_q the q -Jackson derivative (see e.g. [17]), $\mathcal{D}_q P(z) = [P(z) - P(qz)]/[(1-q)z]$.

The big q -Laguerre polynomials $P_n(x; a, c; q)$, are a particular case of the big q -Jacobi polynomials: $P_n(x; a, c; q) = P_n(x; a, 0, c; q)$, therefore all their properties can be obtained from the corresponding ones of the big q -Jacobi by putting $b = 0$. A special case of the big q -Laguerre polynomials are the affine q -Kravchuk polynomials [17, page 101].

The little q -Jacobi polynomials $p_n(x; a, b|q)$ constitute a q -OPS with respect to a linear functional

$$\langle C^{lqJ}, P \rangle := \sum_{s=0}^{\infty} P(s)\rho(s)q^s,$$

Table 3: Parameters of Wall, q -Meixner, and q -Charlier polynomials

	Wall	q -Meixner	q -Charlier
$\sigma(x)$	$q^{-1}x(x-1)$	$q^{-1}c(x-bq)$	$q^{-1}ax$
$\phi(s)$	$-ax$	$(x-1)(x+bc)$	$x(x-1)$
λ_n	$-q^{\frac{3}{2}} \frac{1-q^n}{(1-q)^2 q^n}$	$-q^{\frac{1}{2}} \frac{1-q^n}{(1-q)^2 q^n}$	$q^{\frac{1}{2}} \frac{1-q^n}{(1-q)^2}$
$\rho(x)$	$\frac{(aq;q)_\infty}{(q;q)_s} a^s$	$\frac{(-bcq;q)_\infty}{(-c;q)_\infty} \frac{(bq;q)_s}{(q,-bcq;q)_s} \left(cq^{\frac{s+1}{2}}\right)^s$	$\frac{q^{-\frac{1}{2}}}{(q;q)_s (-a;q)_\infty} \left(aq^{\frac{s+1}{2}}\right)^s$
d_n^2	$\frac{(q;q)_n}{(aq;q)_n} (aq)^n$	$\frac{(q,-c^{-1}q;q)_n}{(bq;q)_n} q^{-n}$	$(-a^{-1}q, q; q)_n q^{-n}$
$P_n(x_0)$	$p_n(0; a q) = 1$	$M_n(1; b, c q) = 1$	$C_n(1; a; q) = 1$
$\hat{\alpha}_n/\alpha_n$	0	$\frac{1-q^n}{1-q}$	$\frac{1-q^n}{1-q}$
$\hat{\gamma}_n/\gamma_n$	$-\frac{1}{(1-q)q^n}$	$\frac{1}{1-q}$	$\frac{1}{1-q}$

where $\rho(s)$ is given in table 2 and it is supported on $[0, 1]$, $0 < a < q^{-1}$, $b < q^{-1}$. Moreover

$$\begin{aligned}
 K_n^{lqJ}(x, 0) &= \frac{(aq, abq; q)_n}{(q, bq; q)_n a^n} \left[\frac{(1 - abq^{n+1})P_n(x; a, b|q) - (x-1)(1-q^{-1})\mathcal{D}_q P_n(q^{-1}x; a, b|q)}{1 - abq} \right], \\
 \mathbb{K}_n^{lqJ}(0) &= \sum_{k=0}^n \frac{(1 - abq^{2k+1})(aq, abq; q)_k}{(1 - abq)(q, bq; q)_k} (aq)^{-k} = \frac{(aq^2, abq^2; q)_n}{(q, bq; q)_n} (aq)^{-n}.
 \end{aligned} \tag{9}$$

The q -Meixner polynomials $M_n(q^{-s}; b, c; q)$ are a q -COP sequence with respect to a linear functional

$$\langle \mathcal{C}^{qM}, P \rangle := \sum_{s=0}^{\infty} P(s) \rho(s) q^{-s},$$

where the weight function $\rho(s)$ is supported on $[1, +\infty)$, $0 < b < q^{-1}$, $0 < c$ (see table 3). Furthermore,

$$\begin{aligned}
 K_n^{qM}(x, 1) &= \frac{(bq; q)_n}{(q, -c^{-1}q; q)_n} \left[M_n(x; b, c; q) - (x+bc)(1-q)\mathcal{D}_{q^{-1}} M_n(x; b, c; q) \right], \\
 \mathbb{K}_n^{qM}(1) &= \sum_{k=0}^n \frac{(bq; q)_k}{(q, -c^{-1}q; q)_k} q^k.
 \end{aligned} \tag{10}$$

A special case of the q -Meixner polynomials are the quantum q -Kravchuk [17, page 98].

The Al-Salam-Carlitz I polynomials $U_n^{(a)}(x; q)$ are orthogonal with respect to the linear functional

$$\langle \mathcal{C}^{ACI}, P \rangle := \int_a^1 P(x) \rho(x) d_q x,$$

where $\rho(x)$ is supported on $[a, 1]$, $a < 0$, $x := x(s) = q^s$. Their main data are in table 3. For these polynomials we have

$$\begin{aligned}
 K_n^{ACI}(x, 1) &= \frac{q^n}{(q; q)_n} \left[U_n^{(a)}(x; q) - (x-a)(1-q^{-1})\mathcal{D}_q U_n^{(a)}(q^{-1}x; q) \right], \\
 \mathbb{K}_n^{ACI}(1) &= \sum_{k=0}^n \frac{1}{(q; q)_k} \left(-aq^{\frac{k-1}{2}} \right)^k.
 \end{aligned} \tag{11}$$

The little q -Laguerre / Wall polynomials $p_n(x; a|q)$ are orthogonal with respect to the linear functional

$$\langle \mathcal{C}^{lqL}, P \rangle := \sum_{s=0}^{\infty} P(s) \rho(s) q^s, \quad x := x(s) = q^s, \quad \text{supp}(\rho) = [0, 1].$$

Since they are a particular case of little q -Jacobi ($b = 0$) all their properties can be obtained from the former ones putting $b = 0$ (see table 3). In particular,

$$K_n^{lqL}(x, 0) = \frac{(aq; q)_n}{(q; q)_n a^n q^n} [p_n(x; a|q) - a(1-q)q^n \mathcal{D}_q p_n(x; a|q)], \quad \mathbb{K}_n^{lqL}(0) = \frac{(aq^2; q)_n}{(q; q)_n (aq)^n}. \quad (12)$$

The q -Laguerre polynomials $L_n^{(\alpha)}(x; q)$ are orthogonal with respect to the linear functional

$$\langle \mathcal{C}^{qL}, P \rangle := \sum_{s=-\infty}^{+\infty} P(cq^k) \rho(s) q^s,$$

where the weight function $\rho(s)$ (see table 2) is supported on $[0, +\infty)$, $a = q^\alpha$, $x := x(s) = cq^s$. In this case

$$K_n^{qL}(x, 0) = q^n L_n^{(\alpha)}(x; q) - \frac{q^{-1} - 1}{a} \mathcal{D}_q L_n^{(\alpha)}(q^{-1}x; q), \quad \mathbb{K}_n^{qL}(0) = \sum_{k=0}^n \frac{(aq; q)_k}{(q; q)_k} q^k = \frac{(aq^2; q)_n}{(q; q)_n}. \quad (13)$$

The q -Charlier polynomials $C_n(q^{-s}; a; q)$ constitute a q -COP sequence with respect to the linear functional

$$\langle \mathcal{C}^{qC}, P \rangle := \sum_{s=0}^{\infty} P(s) \rho(s) q^{-s},$$

where $\rho(s)$ is supported on $[1, +\infty)$, $a > 0$ (see table 3). Moreover,

$$K_n^{qC}(x, 1) = \frac{C_n(x; a; q) - x(1-q) \mathcal{D}_{q^{-1}} C_n(x; a; q)}{(-a^{-1}q, q; q)_n}, \quad \mathbb{K}_n^{qC}(1) = \sum_{k=0}^n \frac{q^k}{(-a^{-1}q, q; q)_k}. \quad (14)$$

The Stieltjes-Wigert polynomials $S_n(x; q)$ correspond to an indeterminate moment problem, so there are infinitely many representations for the linear functional \mathcal{C}^{SW} with respect to which they are orthogonal (see e.g. [17]). Here we will chose the following one

$$\langle \mathcal{C}^{SW}, P \rangle := \int_0^\infty P(x) \rho(x) dx,$$

where $\rho(s)$ is a weight function supported on $[0, +\infty)$ (see table 1). In this case

$$K_n^{SW}(x, 0) = q^n S_n(x; q) - (q^{-1} - 1) \mathcal{D}_q S_n(q^{-1}x; q), \quad \mathbb{K}_n^{SW}(0) = \frac{1}{(q; q)_n}. \quad (15)$$

3 The q -Krall-type orthogonal polynomials

In this section, we will introduce the q -Krall-type orthogonal polynomials. In a very recent paper [9] the authors introduce the “discrete” Krall polynomials as a perturbation of a classical or semiclassical discrete linear functional and they develop a general theory in order to find some algebraic properties such as TTRR, SODE, etc. In this paper we focus our attention on the special case when the starting functional \mathcal{C} is a q -classical functional [21]. Thus we consider the linear functional \mathcal{U} defined as

$$\langle \mathcal{U}, P \rangle = \langle \mathcal{C}, P \rangle + AP(x_0) + BP(x_1), \quad A, B \geq 0, \quad (16)$$

where \mathcal{C} is the linear functional (1) and $x_0, x_1 \in \mathbb{R}$. In [9] a general theory for solving this problem (when N mass points are added) has been presented, nevertheless only two examples were considered in details. Here we will complete this work introducing new examples and we will establish the limit relation among them, in the same way as in [6].

The explicit expression of the polynomials $\tilde{P}_n^{A,B}(s)_q$ orthogonal with respect to the linear functional \mathcal{U} (16) is given by [9] (it is assumed that the polynomials \tilde{P}_n and P_n have the same leading coefficient)

$$\tilde{P}_n(x) = P_n(x) - \sum_{i=1}^M A_i \tilde{P}_n(a_i) K_{n-1}(x, a_i), \quad (17)$$

where $(\tilde{P}_n(a_k))_{k=1}^M$ are the solution of the system

$$\tilde{P}_n(a_k) = P_n(a_k) - \sum_{i=1}^M A_i \tilde{P}_n(a_i) K_{n-1}(a_k, a_i), \quad k = 1, 2, \dots, M.$$

The formula (17) was firstly obtained by Uvarov [23] (see also [15, §2.9]). Hence, the formula [9, Eq. (2.5) page 57] yields

$$\tilde{P}_n^{A,B}(s)_q = P_n(s)_q - \left[\begin{array}{cc} AK_{n-1}(x, x_0) & BK_{n-1}(x, x_1) \\ BK_{n-1}(x, x_1) & 1 + B\mathbb{K}_{n-1}(x_1) \end{array} \right]^t \left[\begin{array}{cc} 1 + A\mathbb{K}_{n-1}(x_0) & BK_{n-1}(x_0, x_1) \\ AK_{n-1}(x_1, x_0) & 1 + B\mathbb{K}_{n-1}(x_1) \end{array} \right]^{-1} \left[\begin{array}{c} P_n(x_0) \\ P_n(x_1) \end{array} \right], \quad (18)$$

where C^t is the transpose of C . Furthermore, the polynomials $\tilde{P}_n^{A,B}(s)_q$ exist for every $n = 0, 1, \dots$ if and only if the following condition

$$\det \left[\begin{array}{cc} 1 + A\mathbb{K}_{n-1}(x_0) & BK_{n-1}(x_0, x_1) \\ AK_{n-1}(x_1, x_0) & 1 + B\mathbb{K}_{n-1}(x_1) \end{array} \right] \neq 0, \quad \forall n \in \mathbb{N}, \quad (19)$$

holds. When the mass $B = 0$ (18) transforms into

$$\tilde{P}_n^A(s)_q = P_n(s)_q - A\tilde{P}_n^A(x_0)_q K_{n-1}(x, x_0), \quad \tilde{P}_n^A(x_0)_q = \frac{P_n(x_0)_q}{1 + A\mathbb{K}_{n-1}(x_0)}. \quad (20)$$

Notice that if $A \geq 0$, then (19) becomes into $1 + A\mathbb{K}_{n-1}(x_0) \geq 1$, hence $\tilde{P}_n^A(s)_q$ exists for every $n = 0, 1, \dots$. The next step is to construct the corresponding families of q -Kral type orthogonal polynomials associated with each family of q -orthogonal polynomials considered in section 2.1. We will start with the big q -Jacobi family since the other families can be obtained from it via taking appropriate limits. Furthermore, we will choose the values of x_0 and x_1 in such a way that the kernels (4) has the simplest form, i.e., (5) and (6).

3.1 The big q -Jacobi-Kral polynomials

Let us consider the linear functional \mathcal{U}^{BqJ} defined by

$$\langle \mathcal{U}^{BqJ}, P \rangle = \langle \mathcal{C}^{BqJ}, P \rangle + AP(x_0) + BP(x_1), \quad A, B \geq 0,$$

where $x_0, x_1 \in \mathbb{R}$ and \mathcal{C}^{BqJ} is the functional (7). The corresponding polynomials will be denoted by $\tilde{P}_n^{A,B}(x; a, b, c; q)$ and constitute a q -analog of the Koornwinder polynomials [18]. The polynomial expression for this family follows from (18)

$$\begin{aligned} \tilde{P}_n^{A,B}(x; a, b, c; q) = & P_n(x; a, b, c; q) - \left[\begin{array}{cc} AK_{n-1}^{BqJ}(x; x_0) & BK_{n-1}^{BqJ}(x; x_1) \end{array} \right] \times \\ & \left[\begin{array}{cc} 1 + A\mathbb{K}_{n-1}^{BqJ}(x_0) & BK_{n-1}^{BqJ}(x_0, x_1) \\ AK_{n-1}^{BqJ}(x_1, x_0) & 1 + B\mathbb{K}_{n-1}^{BqJ}(x_1) \end{array} \right]^{-1} \left[\begin{array}{c} P_n(x_0; a, b, c; q) \\ P_n(x_1; a, b, c; q) \end{array} \right]. \end{aligned}$$

Now, we are going to consider two specific cases:

1. The q -Koornwinder polynomials obtained when we add two mass points at the endpoints of the interval of orthogonality of the big q -Jacobi polynomials. i.e., $x_0 = cq$ and $x_1 = aq$. For these values,

$$\tilde{P}_n^{A,B}(x; a, b, c; q) := P_n(x; a, b, c; q) - A\tilde{P}_n^{A,B}(cq)K_{n-1}^{BqJ}(x, cq) - B\tilde{P}_n^{A,B}(aq)K_{n-1}^{BqJ}(x, aq).$$

Then, using (8) and taking into account the identities [17, Eq. (3.5.6), (3.5.7)] for the big q -Jacobi polynomials,

$$\begin{aligned} P_n(x; a, b, c; q) - P_n(qx; a, b, c; q) &= \frac{q(q^{-n} - 1)(1 - abq^{n+1})}{(1 - aq)(1 - cq)} xP_{n-1}(qx; aq, bq, cq; q), \\ \mathcal{D}_q P_n(x; a, b, c; q) &= \frac{q(q^{-n} - 1)(1 - abq^{n+1})}{(1 - q)(1 - aq)(1 - cq)} P_{n-1}(qx; aq, bq, cq; q), \end{aligned} \quad (21)$$

we get

$$\tilde{P}_n^{A,B}(x; a, b, c; q) = P_n(x; a, b, c; q) - A_{n-1}P_{n-1}(x; a, b, c; q) - B_{n-1}(x)P_{n-2}(x; aq, bq, cq; q), \quad (22)$$

where

$$A_n = \frac{(abq^2; q)_n}{(q; q)_n} \left(A\tilde{P}_{n+1}^{A,B}(cq) \frac{(cq; q)_n}{c^n(abc^{-1}q; q)_n} + B\tilde{P}_{n+1}^{A,B}(aq) \frac{(aq; q)_n}{a^n(bq; q)_n} \right),$$

and

$$B_n(x) = \frac{(abq^2; q)_n(1 - q^n)}{(1 - aq)(1 - cq)(q; q)_n q^n} \left(A\tilde{P}_{n+1}^{A,B}(cq) \frac{(cq; q)_n(x - aq)}{c^n(abc^{-1}q; q)_n} + B\tilde{P}_n^{A,B}(aq) \frac{(aq; q)_n(x - cq)}{a^n(bq; q)_n} \right).$$

Before analyzing the following particular case let us show that these polynomials can be written as a basic hypergeometric series. In fact, by definition of the big q -Jacobi polynomials and (22) we obtain

$$\begin{aligned} \tilde{P}_n^{A,B}(x; a, b, c; q) &= \sum_{k=0}^{\infty} \frac{(q^{-n}, abq^n, x; q)_k q^k}{(aq^2, cq^2, q; q)_k} \left(\frac{(1 - aq^{k+1})(1 - cq^{k+1})}{(1 - aq)(1 - cq)} \left[\frac{1 - abq^{n+k}}{1 - abq^n} \right. \right. \\ &\quad \left. \left. + A_n \frac{1 - q^{-n+k}}{1 - q^{-n}} \right] + B_n(x) \frac{(1 - q^{-n+k})(1 - q^{-n+k+1})}{(1 - q^{-n})(1 - q^{-n+1})} \right). \end{aligned} \quad (23)$$

Now, if we use the identity $(q^{\alpha+1}; q)_m(1 - q^\alpha) = (q^\alpha; q)_m(1 - q^{\alpha+m})$ as well as the fact that the polynomial on q^k at the RHS, namely π_3 , has three zeros, namely q^{α_1} , q^{α_2} , and q^{α_3} which depend, in general, of all parameters, i.e. $\alpha_{1,2,3} := \alpha_{1,2,3}(n, x; a, b, c, A, B; q)$, and

$$\pi_3(q^k) = r(x)(q^k - q^{\alpha_1})(q^k - q^{\alpha_2})(q^k - q^{\alpha_3}), \quad \deg r(x) = 1.$$

Then we get

$$\tilde{P}_n^{A,B}(x; a, b, c; q) = \tilde{D}_n(x) {}_6\varphi_5 \left(\begin{matrix} q^{-n}, abq^n, q^{1-\alpha_1}, q^{1-\alpha_2}, q^{1-\alpha_3}, x \\ aq^2, cq^2, q^{-\alpha_1}, q^{-\alpha_2}, q^{1-\alpha_3} \end{matrix} \middle| q; q \right), \quad (24)$$

where

$$\tilde{D}_n(x) = (1 - q^{\alpha_1})(1 - q^{\alpha_2})(1 - q^{\alpha_3})r(x).$$

Remark 3.1 Notice that \tilde{D}_n is, in general, a polynomial of degree 1 in x . To see that $\tilde{P}_n^{A,B}$ is a polynomial of degree n we only need to evaluate (23) for $k = n$ since in this case the second term on the last bracket vanishes.

2. The case $x_0 = aq$ and $x_1 = 1$. For these values,

$$\hat{P}_n^{A,B}(x; a, b, c; q) := P_n(x; a, b, c; q) - A\hat{P}_n^{A,B}(aq)K_{n-1}^{BqJ}(x, aq) - B\hat{P}_n^{A,B}(1)K_{n-1}^{BqJ}(x, 1).$$

Then, using (8) and the relation (21), we get

$$\begin{aligned} \widehat{P}_n^{A,B}(x; a, b, c; q) = & P_n(x; a, b, c; q) - \widehat{A}_{n-1}P_{n-1}(x; a, b, c; q) \\ & - \widehat{B}_{n-1}(x)P_{n-2}(x; aq, bq, cq; q) - \widehat{C}_{n-1}(x)P_{n-2}(qx; aq, bq, cq; q) \end{aligned} \quad (25)$$

where

$$\begin{aligned} \widehat{A}_n &= \frac{(aq, abq^2; q)_n}{(q, bq; q)_n a^n} \left(A\widehat{P}_{n+1}^{A,B}(aq) + \frac{(-1)^n (cq; q)_n}{(abc^{-1}q; q)_n c^n q^{n^2/2+3n/2}} B\widehat{P}_{n+1}^{A,B}(1) \right), \\ \widehat{B}_n(x) &= \frac{(aq, abq^2; q)_n (1 - q^n)(x - cq)}{(q, bq; q)_n (aq)^n (1 - aq)(1 - cq)} A\widehat{P}_{n+1}^{A,B}(aq), \\ \widehat{C}_n(x) &= \frac{(-1)^n (aq, abq^2, cq; q)_n (1 - q^n) aq^3 (bx - c)}{(q, bq, abc^{-1}q; q)_n a^n c^n q^{n^2/2+3n/2} (1 - aq)(1 - cq)} B\widehat{P}_{n+1}^{A,B}(1). \end{aligned}$$

Hence, following the same idea of the previous case we get, this family admits another representation in terms of basic hypergeometric series. Indeed some calculations, in the same fashion as in the previous case, yield

$$\widehat{P}_n^{A,B}(x; a, b, c; q) = \widehat{D}_n(x) {}_6\varphi_5 \left(\begin{matrix} q^{-n}, abq^n, q^{1-\beta_1}, q^{1-\beta_2}, q^{1-\beta_3}, x \\ aq^2, cq^2, q^{-\beta_1}, q^{-\beta_2}, q^{-\beta_3} \end{matrix} \middle| q; q \right), \quad (26)$$

where $\widehat{D}_n(x)$ depends of the parameters defined for this family, and q^{β_1} , q^{β_2} , and q^{β_3} are the zeros of a certain cubic polynomial on q^k , $\beta_i := \beta_i(n, x; a, b, c, A, B; q)$, $i = 1, 2, 3$, obtained as before from the expression (25) and the basic series representation of the big q -Jacobi polynomials.

Remark 3.2 *Observe that, although \widehat{D}_n is a polynomial of degree 2 in x , $\widetilde{P}_n^{A,B}$ is a polynomial of degree n in x since the evaluation of the basic hypergeometric series (26) for both $k = n - 1$ and $k = n$ vanishes.*

Two particular interesting cases are the following: Setting $A = 0$ in the q -Koornwinder polynomials (24) we obtain

$$\widetilde{P}^B(x; a, b, c; q) = \widetilde{D}_n^B(x) {}_6\varphi_5 \left(\begin{matrix} q^{-n}, abq^n, q^{1-\alpha_1}, q^{1-\alpha_2}, q^{1-\alpha_3}, x \\ aq^2, cq^2, q^{-\alpha_1}, q^{-\beta_2}, q^{-\beta_3} \end{matrix} \middle| q; q \right),$$

and setting $A = 0$ in the second family (26) we get

$$\widehat{P}^B(x; a, b, c; q) = \widehat{D}_n^B(x) {}_6\varphi_5 \left(\begin{matrix} q^{-n}, abq^n, q^{1-\beta_1}, q^{1-\beta_2}, q^{1-\beta_3}, x \\ aq^2, cq^2, q^{-\beta_1}, q^{-\beta_2}, q^{-\beta_3} \end{matrix} \middle| q; q \right).$$

Setting in all the above formulas $c = q^{-N-1}$ we obtain the q -Hahn-Krall polynomials.

Before continuing let us point out that the above families satisfy a three-term recurrence relation and a second order linear difference equation. For more details see [9].

3.2 Examples adding one mass point

3.2.1 The big q -Laguerre-Krall polynomials

It is a particular case of the q -Krall big q -Jacobi. In this case the linear functional \mathcal{U}^{BqL} is

$$\langle \mathcal{U}^{BqL}, P \rangle = \langle \mathcal{C}^{BqL}, P \rangle + AP(aq), \quad A \geq 0,$$

where \mathcal{C}^{BqL} is the functional with respect the big q -Laguerre are orthogonal. The explicit expression for the polynomials is

$$\widetilde{P}_n^A(x; a, c; q)_q = P_n(x; a, c; q)_q - A \frac{P_n(aq; a, c; q)_q K_{n-1}^{BqL}(x, aq)}{1 + AK_{n-1}^{BqL}(aq)},$$

or, equivalently, putting $B = 0$ in (22) and set $b = 0$,

$$\begin{aligned} \tilde{P}_n^A(x; a, c; q) &= P_n(x; a, c; q)_q - \frac{A(-cq^{\frac{n+1}{2}})^n (aq; q)_{n-1}}{1 + A\mathbb{K}_{n-1}^{BqL}(aq)} \\ &\times \left[\frac{P_{n-1}(x; a, c; q)_q - (x - cq)(1 - q^{-1})\mathcal{D}_q P_{n-1}(q^{-1}x; a, c; q)_q}{(cq; q)_n (q; q)_{n-1} a^{n-1}} \right] \end{aligned}$$

They can be represented as a ${}_6\varphi_5$ basic series.

3.2.2 The little q -Jacobi-Krall polynomials

These polynomials are orthogonal with respect to the linear functional \mathcal{U}^{lqJ}

$$\langle \mathcal{U}^{lqJ}, P \rangle = \langle \mathcal{C}^{lqJ}, P \rangle + AP(0), \quad A \geq 0,$$

where \mathcal{C}^{lqJ} is the functional of little q -Jacobi polynomials. The representation formulas for this family is (see (17), (9))

$$\begin{aligned} \tilde{p}_n^A(x; a, c|q)_q &= p_n(x; a, c|q)_q - A \frac{p_n(0; a, c|q)_q K_{n-1}^{lqJ}(x, 0)}{1 + A\mathbb{K}_{n-1}^{lqJ}(0)} = p_n(x; a, c|q)_q - \frac{A(aq, abq; q)_{n-1}}{1 + A\mathbb{K}_{n-1}^{lqJ}(0)} \\ &\times \left[\frac{(1 - abq^n)P_{n-1}(x; a, b|q)_q - (x - 1)(1 - q^{-1})\mathcal{D}_q P_{n-1}(q^{-1}x; a, b|q)_q}{(1 - abq)(q, bq; q)_{n-1} a^{n-1}} \right]. \end{aligned}$$

This case leads to a ${}_6\varphi_5$ basic series.

3.2.3 The q -Meixner-Krall polynomials

These polynomials are orthogonal with respect to the linear functional \mathcal{U}^{qM}

$$\langle \mathcal{U}^{qM}, P \rangle = \langle \mathcal{C}^{qM}, P \rangle + AP(1), \quad A \geq 0,$$

where \mathcal{C}^{qM} is the functional of the q -Meixner polynomials. The explicit expression for this family is (see (17), (10))

$$\begin{aligned} \tilde{M}_n^A(x; a, b; q)_q &= M_n(x; a, b|q)_q - A \frac{M_n(1; a, b|q)_q K_{n-1}^{qM}(x, 1)}{1 + A\mathbb{K}_{n-1}^{qM}(1)} = M_n(x; a, b|q)_q - \frac{A(bq; q)_{n-1}}{1 + A\mathbb{K}_{n-1}^{qM}(1)} \\ &\times \left[\frac{M_{n-1}(x; b, c; q) - (x + bc)(1 - q)\mathcal{D}_{q^{-1}} M_{n-1}(x; b, c; q)}{(q, -c^{-1}q; q)_{n-1}} \right]. \end{aligned}$$

And, this case leads to a ${}_5\varphi_4$ basic series.

3.2.4 The Al-Salam-Carlitz-Krall I polynomials

These polynomials are orthogonal with respect to the linear functional \mathcal{U}^{ACI}

$$\langle \mathcal{U}^{ACI}, P \rangle = \langle \mathcal{C}^{ACI}, P \rangle + AP(1), \quad A \geq 0,$$

where \mathcal{C}^{ACI} is the functional of the Al Salam Carlitz I polynomials. The representation formula for this family is (see (17), (11))

$$\begin{aligned} \tilde{U}_n^{(a), A}(x; q)_q &= U_n^{(a)}(x; q)_q - A \frac{U_n^{(a)}(1; q)_q K_{n-1}^{ACI}(x, 1)}{1 + A\mathbb{K}_{n-1}^{ACI}(1)} = U_n^{(a)}(x; q)_q - A \frac{U_n^{(a)}(1; q)_q}{1 + A\mathbb{K}_{n-1}^{ACI}(1)} \\ &\times \frac{q^{n-1}}{(q; q)_{n-1}} \left[U_{n-1}^{(a)}(x; q) - (x - a)(1 - q^{-1})\mathcal{D}_q U_{n-1}^{(a)}(q^{-1}x; q) \right]. \end{aligned}$$

This case leads to a ${}_5\varphi_4$ basic series. This family was considered in [9]. Since the Al-Salam-Carlitz II are related with the Al-Salam-Carlitz I by the change $q \rightarrow q^{-1}$ the corresponding q -Krall family can be obtained by the same change.

3.2.5 The little q -Laguerre-Krall/Wall-Krall polynomials

These polynomials are orthogonal with respect to the linear functional \mathcal{U}^{lqL}

$$\langle \mathcal{U}^{lqL}, P \rangle = \langle \mathcal{C}^{lqL}, P \rangle + AP(0), \quad A \geq 0,$$

where \mathcal{C}^{lqL} is the functional of the q -Laguerre/Wall polynomials. The explicit expression for this family is (see (17), (12))

$$\begin{aligned} \tilde{p}_n^A(x; a|q)_q &= p_n(x; a|q)_q - \frac{p_n(0; a|q)_q K_{n-1}^{lqL}(x, 0)}{1 + A\mathbb{K}_{n-1}^{lqL}(0)} = p_n(x; a|q)_q - \frac{A}{1 + A\mathbb{K}_{n-1}^{lqL}(0)} \\ &\times \frac{(aq; q)_{n-1}}{(q; q)_{n-1} a^{n-1} q^{n-1}} [p_{n-1}(x; a|q) - a(1-q)q^{n-1} \mathcal{D}_q p_{n-1}(x; a|q)]. \end{aligned}$$

This case leads to a ${}_4\varphi_3$ basic series.

3.2.6 The q -Laguerre-Krall polynomials

These polynomials are orthogonal with respect to the linear functional \mathcal{U}^{qL}

$$\langle \mathcal{U}^{qL}, P \rangle = \langle \mathcal{C}^{qL}, P \rangle + AP(0), \quad A \geq 0,$$

where \mathcal{C}^{qL} is the functional of the q -Laguerre polynomials. In this case (17) and (13) yield

$$\begin{aligned} \tilde{L}_n^{(\alpha), A}(x; q)_q &= L_n^{(\alpha)}(x; q)_q - A \frac{L_n^{(\alpha)}(0; q)_q K_{n-1}^{qL}(x, 0)}{1 + A\mathbb{K}_{n-1}^{qL}(0)} \\ &= L_n^{(\alpha)}(x; q)_q - A \frac{L_n^{(\alpha)}(0; q)_q}{1 + A\mathbb{K}_{n-1}^{qL}(0)} \left[q^{n-1} L_{n-1}^{(\alpha)}(x; q) - \frac{q^{-1} - 1}{a} \mathcal{D}_q L_{n-1}^{(\alpha)}(q^{-1}x; q) \right]. \end{aligned}$$

This case leads to a ${}_4\varphi_3$ basic series.

3.2.7 The q -Charlier-Krall polynomials

These polynomials are orthogonal with respect to the linear functional \mathcal{U}^{qC}

$$\langle \mathcal{U}^{qC}, P \rangle = \langle \mathcal{C}^{qC}, P \rangle + AP(1), \quad A \geq 0,$$

where \mathcal{C}^{qC} is the weight function of the q -Charlier polynomials. For these polynomials (17) and (14) yield

$$\begin{aligned} \tilde{C}_n^A(x; a; q)_q &= C_n(x; a; q)_q - A \frac{C_n(1; a; q)_q K_{n-1}^{qC}(x, 1)}{1 + A\mathbb{K}_{n-1}^{qC}(1)} \\ &= C_n(x; a; q)_q - \frac{A}{1 + A\mathbb{K}_{n-1}^{qC}(1)} \left[\frac{C_{n-1}(x; a; q) - x(1-q) \mathcal{D}_{q^{-1}} C_{n-1}(x; a; q)}{(-a^{-1}q, q; q)_{n-1}} \right]. \end{aligned}$$

This case leads to a ${}_4\varphi_3$ basic series.

3.2.8 The Stieltjes-Wigert-Krall polynomials

These polynomials are orthogonal with respect to the linear functional \mathcal{U}^{SW}

$$\langle \mathcal{U}^{SW}, P \rangle = \langle \mathcal{C}^{SW}, P \rangle + AP(0), \quad A \geq 0,$$

where \mathcal{C}^{SW} is the functional of the Stieltjes-Wigert polynomials. The representation formula for this family has the form (see (17), (15))

$$\tilde{S}_n^A(x; q)_q = S_n(x; q)_q - A \frac{S_n(0; q)_q}{1 + A\mathbb{K}_{n-1}^{SW}(0)} [q^{n-1} S_{n-1}(x; q) - (q^{-1} - 1) \mathcal{D}_q S_{n-1}(q^{-1}x; q)].$$

This case leads to a ${}_3\varphi_3$ basic series. This family was firstly studied in [12].

3.3 Some algebraic properties of $\tilde{P}_n^A(s)_q$

In [9] it is shown that the q -Krall-type orthogonal polynomials satisfy a second order linear difference equation of the form

$$\tilde{\sigma}(s; n)\tilde{P}_n(s-1)_q - \tilde{\varphi}(s; n)\tilde{P}_n(s)_q + \tilde{\zeta}(s; n)\tilde{P}_n(s+1)_q = 0, \quad (27)$$

where

$$\begin{aligned} \tilde{\sigma}(s; n) &= t(s; n)[a(s; n)d(s; n) - c(s; n)b(s; n)], \\ \tilde{\varphi}(s; n) &= -\pi(s; n)[c(s; n)f(s; n) - e(s; n)d(s; n)], \\ \tilde{\zeta}(s; n) &= r(s; n)[e(s; n)b(s; n) - a(s; n)f(s; n)], \end{aligned} \quad (28)$$

being

$$\begin{aligned} r(s; n) &= \zeta(s+1; n)\pi(s+1; n), \quad c(s; n) = -\sigma(s+1; n)b(s+1; n), \\ d(s; n) &= a(s+1; n)\zeta(s+1; n) + b(s+1; n)\varphi(s+1; n), \quad t(s; n) = \sigma(s; n)\pi(s-1; n), \\ e(s; n) &= \sigma(s; n)b(s-1; n) + a(s-1; n)\varphi(s; n), \quad f(s; n) = -a(s-1; n)\zeta(s; n), \end{aligned}$$

where $a(s; n)$, $b(s; n)$, and $\pi(s; n)$ are the coefficients of the representation formula for the Krall-type polynomials \tilde{P}_n

$$\pi(s; n)\tilde{P}_n(s)_q = a(s; n)P_n(s)_q + b(s; n)P_n(s+1)_q,$$

and σ , φ , and ζ are the coefficients of the second order difference equation that the starting polynomials P_n satisfy

$$\sigma(s; n)P_n(s-1)_q - \varphi(s; n)P_n(s)_q + \zeta(s; n)P_n(s+1)_q = 0.$$

Also in [9] the TTRR for the polynomials $\tilde{P}_n^A(s)_q$ is computed

$$x(s)\tilde{P}_n^A(s)_q = \alpha_n^A \tilde{P}_{n+1}^A(s)_q + \beta_n^A \tilde{P}_n^A(s)_q + \gamma_n^A \tilde{P}_{n-1}^A(s)_q, \quad (29)$$

where the coefficients α_n^A , β_n^A , and γ_n^A are given by

$$\begin{aligned} \alpha_n^A &= \alpha_n, \quad \gamma_n^A = \alpha_{n-1} \frac{\tilde{d}_n^2}{d_{n-1}^2} \neq 0, \\ \beta_n^A &= \beta_n + \frac{AP_n(s_0)_q}{d_n^2} \left(\alpha_n \frac{P_{n+1}(s_0)_q}{1 + A\mathbb{K}_n(s_0)} - \gamma_n \frac{P_{n-1}(s_0)_q}{1 + A\mathbb{K}_{n-1}(s_0)} \right), \end{aligned}$$

where α_n , β_n , and γ_n are the coefficients of the TTRR of the starting family of q -polynomials (2), $d_n^2 = \langle \mathcal{C}, P_n P_n \rangle$ and

$$\tilde{d}_n^2 = \langle \mathcal{U}, \tilde{P}_n^A \tilde{P}_n^A \rangle = d_n^2 + \left[A\tilde{P}_n^A(s_0)_q \right]^2 \mathbb{K}_{n-1}(s_0) + A \left[\tilde{P}_n^A(s_0)_q \right]^2 = \frac{1 + A\mathbb{K}_n(s_0)}{1 + A\mathbb{K}_{n-1}(s_0)} d_n^2,$$

are the square of the norms of the polynomials P_n and \tilde{P}_n^A , respectively.

3.3.1 Some examples

Here we will restrict ourselves to the more simple cases. The other cases are analogously and we will omit them here.

Little q -Laguerre-Krall / Wall-Krall polynomials: The Wall polynomials satisfy the following SODE (recall that $x := q^s$)

$$ap_n(qx; a|q) + (1-x)p_n(x/q; a|q) + [(x-a-1) + q^{-n}(1-q^n)x]p_n(x; a|q) = 0$$

as well as the relation [21, Eq. (3.2), page 175]

$$x\mathcal{D}_q p_n(x; a|q) = \frac{1-q^n}{1-q} (p_n(x; a|q) - p_{n-1}(x; a|q)) \Rightarrow (1-q^n)p_{n-1}(x; a|q) = -q^n p_n(x; a|q) + p_n(qx; a|q).$$

Then, for the kernel $K_{n-1}(x, 0)$ we have the expression

$$xK_{n-1}(x, 0) = -\frac{(aq; q)_{n+1}}{(1-q^n)(q; q)_n a^n} [p_n(x; a|q) - p_n(qx; a|q)].$$

Thus from the expression for the Wall-Krall polynomials in section 3.2.5 we obtain

$$x\tilde{p}_n^A(x; a|q)_q = (x + b_n^A)p_n(x; a|q)_q - b_n^A p_n(qx; a|q)_q,$$

$$x\tilde{p}_n^A(x; a|q)_q = \left(x - \frac{c_n}{q^{n-1}}\right)p_n(x; a|q) + ac_n \left(\frac{1-q^{n-1}}{q^{n-1}} - \frac{1+a(1-q^{n-1}-q^n)}{a}\right) \frac{(q^n p_n(x; a|q) - p_n(qx; a|q))}{(1-q^n)(1-aq^n)}$$

where

$$c_n = \frac{A(aq; q)_{n-1}}{(1 + A\mathbb{K}_{n-1}(0))a^{n-1}(q; q)_{n-1}}.$$

where

$$b_n^A = A \frac{p_n(0; a|q)_q}{1 + A\mathbb{K}_{n-1}^{Lq}(0)} \frac{(aq; q)_{n+1}}{(1-q^n)(q; q)_n a^n}.$$

Therefore, Theorem 2 of [9, page 60-61] gives the following values for the coefficients of the SODE (27)

$$\begin{aligned} \zeta(s; n) &= -\frac{a(x+1)(q^n x(b_n^A)^2 - x(b_n^A)^2 - x^2 b_n^A - aq^n x b_n^A + q^n x b_n^A + x b_n^A - q^n b_n^A - aq^n x^2 + aq^n x)}{q^n}, \\ \tilde{\varphi}(s; n) &= x \left[((1-q^{-n})x - x + a + 1)(b_n^A + x - 1) - (1-x)b_n^A \right] \left(a(b_n^A + x + 1) - ((1-q^{-n})(x+1) - x + a)b_n^A \right) \\ &\quad + ax^2(-b_n^A - x + 1)b_n^A, \\ \tilde{\sigma}(s; n) &= \frac{(x-1)^2(q^n x(b_n^A)^2 - x(b_n^A)^2 + q^n(b_n^A)^2 - (b_n^A)^2 - x^2 b_n^A - aq^n x b_n^A + q^n x b_n^A - x b_n^A - aq^n b_n^A - aq^n x^2 - aq^n x)}{q^n}. \end{aligned}$$

For this family we have the following coefficients of the TTRR (29)

$$\begin{aligned} \alpha_n^A &= -q^n(1 - aq^{n+1}), \\ \beta_n^A &= q^n(1 - aq^{n+1}) + aq^n(1 - q^n) \\ &\quad - A \frac{q^n(1 - aq^{n+1})(1 - aq)}{(1 - aq)d_n^2 + A(1 - aq^{n+1})} + A \frac{q^{n-1}(1 - aq^n)(1 - aq)}{(1 - aq)d_{n-1}^2 + A(1 - aq^n)}, \\ \gamma_n^A &= \alpha_{n-1}^A \frac{(d_n^2((1 - aq)d_{n-1}^2 + 1 - aq^n) + (1 - aq)d_{n-1}^2)((1 - aq)d_{n-2}^2 + 1 - aq^{n-1})}{(d_{n-1}^2((1 - aq)d_{n-2}^2 + 1 - aq^{n-1}) + (1 - aq)d_{n-2}^2)((1 - aq)d_{n-1}^2 + 1 - aq^n)}, \end{aligned}$$

where $d_n^2 = \frac{(q; q)_n}{(aq; q)_n} (aq)^n$.

q -Stieltjes-Wigert-Krall polynomials: The q -Stieltjes-Wigert polynomials satisfy the SODE

$$xS_n(qx; q) + S_n(x/q; q) - [1 + q^n x]S_n(x; q) = 0.$$

Now, combining relation [21, Eq. (3.2), page 175] and the TTRR of the Stieltjes-Wigert polynomials we have

$$\begin{aligned} x^2\mathcal{D}_q S_n(x; q) &= \frac{1-q^n}{1-q} (q^n x + q^2)q^{-n} s_n(x; q) - \frac{q^{-n}}{1-q} S_{n-1}(x; q) \Rightarrow \\ S_{n-1}(x; q) &= -[q^{2n}x + q^2(q^n - 1)]S_n(x; q) - xq^n S_n(qx; q). \end{aligned}$$

Then, for the kernel $K_{n-1}(x, 0)$ we have the expression

$$xK_{n-1}(x, 0) = -(1 - q^{n+1})q^{-3n-1} \left[((1 - q^n)(1 - q^2) + q^{2n}x)S_n(x; q) - xq^n S_n(qx; q) \right].$$

Thus from the expression for the q -Stieltjes-Wigert-Krall polynomials in section 3.2.8 we obtain

$$x\tilde{S}_n^A(x; q)_q = \left[(1 + b_n^A q^{2n})x + (1 - q^n)(1 - q^2)b_n^A \right] S_n(x; a)_q + b_n^A q^n x S_n(qx; q)_q,$$

being

$$b_n^A = Aq^{-3n-1} \frac{S_n(0; q)_q(1 - q^{n+1})}{1 + AK_{n-1}^{SW}(0)}.$$

Therefore, Theorem 2 of [9, page 60-61] gives the following values (28) for the coefficients of the SODE (27) which explicit expression we will omit here.

For this family we have the following coefficients of the TTRR (29)

$$\begin{aligned} \alpha_n^A &= -(1 - q^{n+1})q^{-2n-1}, \\ \beta_n^A &= (1 + q - q^{n+1})q^{-2n-1} - \frac{A}{q^{3n+1}} \left(\frac{1}{(q; q)_n + A} - \frac{q^3}{(q; q)_{n-1} + A} \right), \\ \gamma_n^A &= -\frac{1}{q^{2n+2}} \frac{((q; q)_n + A)((q; q)_{n-1} + A(1 - q^{n-1}))}{((q; q)_{n-1} + A)((q; q)_n + A(1 - q^n))}. \end{aligned}$$

4 Limit relations between q -Krall-type orthogonal polynomials

In this section, we study the limit relations involving the q -Krall-type orthogonal polynomials associated with some families of q -polynomials of the q -Hahn Tableau [19, 21]. As we already pointed out the q -Koornwinder polynomials $\tilde{P}_n^{A,B}(x; a, b, c; q)$ (22) is the q -analogue of the Koornwinder polynomials $P_n^{A,B}(x)$ [18]. In fact, a direct calculation show

$$\lim_{q \rightarrow 1^-} \tilde{P}_n^{A,B}(x; a, b, c; q) = P_n^{A,B}(x).$$

Let now consider the other limits.

1. **Big q -Jacobi** \rightarrow **Big q -Laguerre**. We know that the big q -Laguerre is a special case of big q -Jacobi setting $b = 0$, i.e. $P_n(x; a, 0, c; q) = P_n(x; a, c; q)$. Then, from (20) we get

$$\tilde{P}_n^A(x; a, 0, c; q) = \tilde{P}_n^A(x; a, c; q).$$

2. **Big q -Jacobi** \rightarrow **Little q -Jacobi**. The little q -Jacobi polynomials can be obtained from the big q -Jacobi polynomials by linear change of the variable $x \rightarrow cqx$ and taking the limit $c \rightarrow \infty$, i.e. $\lim_{c \rightarrow \infty} P_n(cqx; a, b, c; q) = p_n(x; a, b|q)$. In this case, putting $xcq = aq$ and taking the limit $c \rightarrow \infty$ we get $x \rightarrow 0$, thus $\lim_{c \rightarrow \infty} P_n(aq; a, b, c; q) = p_n(0; a, b|q)$. Taking into account that the the norm of big q -Jacobi transforms into the norm of the little q -Jacobi we obtain

$$\lim_{c \rightarrow \infty} \tilde{P}_n^A(cqx; a, b, c; q) = \tilde{p}_n^A(x; a, b|q).$$

3. **Big q -Jacobi** \rightarrow **q -Meixner**. If we take the limit $a \rightarrow \infty$ in the big q -Jacobi we obtain the q -Meixner polynomials [17]. Thus, from (20) we deduce

$$\lim_{a \rightarrow \infty} \tilde{P}_n^A(q^{-s}; a, b, c; q) = \tilde{M}_n^A(q^{-s}; c, -b^{-1}; q).$$

4. **Big q -Jacobi \rightarrow Hahn.** Setting $c = q^{-N-1}$ in the big q -Jacobi we get the q -Hahn polynomials $\widehat{P}_n^{A,B}(x; a, b, q^{-N-1}; q) = \widehat{Q}_n^{A,B}(x; a, b, N|q)$. Substituting $x = q^{-x}$, $a = q^\alpha$, $b = q^\beta$, we recover the Hahn-Krall polynomials studied in [5] $\lim_{q \rightarrow 1^-} \widehat{Q}_n^{0,A}(q^{-x}; q^\alpha, q^\beta, q^{-N-1}|q) = Q_n^{0,A}(x; \alpha, \beta, N)$. Notice that from the Hahn-Krall polynomials it is possible to obtain several other families of Krall-type polynomials via appropriate limits (see [6]).

5. **Big q -Laguerre \rightarrow Al-Salam-Carlitz I.** Substituting $x \rightarrow aqx$ and $c \rightarrow ac$ in the big q -Laguerre polynomials and taking $a \rightarrow 0$ we obtain the Al-Salam-Carlitz I polynomials $\lim_{a \rightarrow 0} \frac{P_n(aqx; a, ac; q)}{a^n} = q^n U_n^{(c)}(x; q)$. Therefore,

$$\lim_{a \rightarrow 0} \frac{\widetilde{P}_n^A(aqx; a, ac; q)}{a^n} = q^n \widetilde{U}_n^{(c),A}(x; q).$$

6. **Big q -Laguerre \rightarrow Little q -Laguerre/Wall.** The little q -Laguerre polynomials can be obtained from the big q -Laguerre polynomials by setting $x \rightarrow bqx$ and then taking the limit $b \rightarrow \infty$: $\lim_{b \rightarrow \infty} P_n(bqx; a, b; q) = p_n(x; a|q)$. Thus

$$\lim_{b \rightarrow \infty} \widetilde{P}_n^A(bqx; a, b; q) = \widetilde{p}_n^A(x; a|q).$$

7. **Little q -Jacobi \rightarrow Little q -Laguerre/Wall.** Setting $b = 0$ in the little q -Jacobi polynomials we get the little q -Laguerre $p_n(x; a, 0|q) = p_n(x; a|q)$, then

$$\widetilde{p}_n^A(x; a, 0|q) = \widetilde{p}_n^A(x; a|q).$$

8. **Little q -Jacobi \rightarrow q -Laguerre.** In this case straightforward calculations give us

$$\lim_{b \rightarrow \infty} \widetilde{p}_n^A\left(-\frac{x}{bq}; q^\alpha, b|q\right) = \frac{(q; q)_n}{(q^{\alpha+1}; q)_n} \widetilde{L}_n^{(\alpha),A}(x; q).$$

9. **q -Meixner \rightarrow q -Laguerre.** Straightforward calculations yield

$$\lim_{c \rightarrow \infty} \widetilde{M}_n^A(cax; a, c; q) = \frac{(q; q)_n}{(q^{\alpha+1}; q)_n} \widetilde{L}_n^{(\alpha),A}(x; q).$$

10. **q -Meixner \rightarrow q -Charlier.** $\lim_{b \rightarrow 0} \widetilde{M}_n^A(x; b, a; q) = \widetilde{C}_n^A(x; a; q)$.

11. **q -Laguerre \rightarrow Stieltjes-Wigert.** $\lim_{\alpha \rightarrow \infty} \widetilde{L}_n^{(\alpha),A}(xq^{-\alpha}; q) = \widetilde{S}_n^A(x; q)$.

12. **q -Charlier \rightarrow Stieltjes-Wigert.** $\lim_{a \rightarrow \infty} \widetilde{C}_n^A(ax; a; q) = (q; q)_n \widetilde{S}_n^A(x; q)$.

To finish this work let us point out that for the other families of the q -Hahn tableau, i.e., for the q -Kravchuk, alternative q -Charlier the same results can be obtained in an analogous way.

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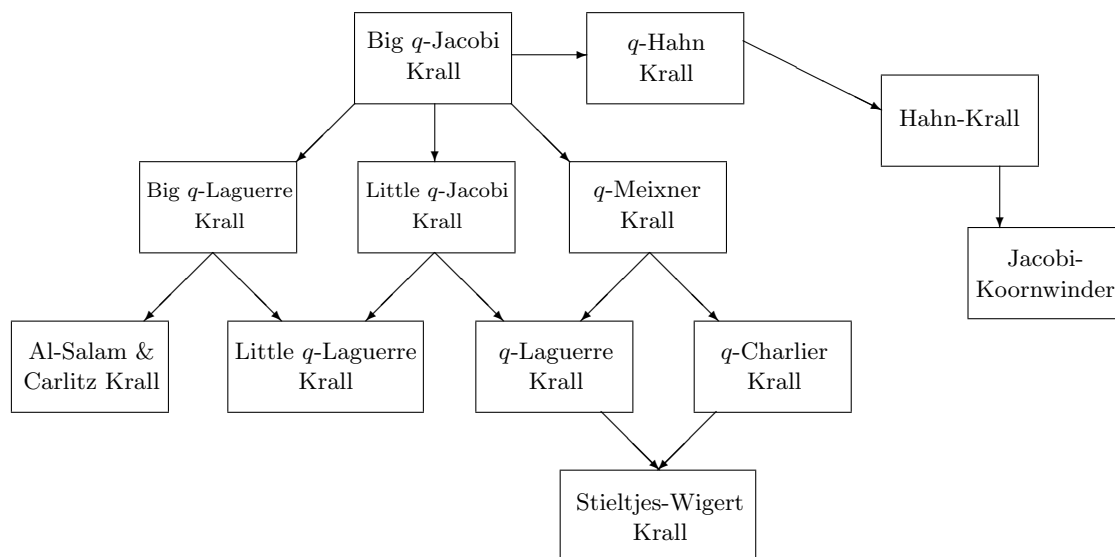


Figure 1: The q -Hahn-Krall Tableau

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