ORTHONET 2013
PROBLEM SESSION

ANDRÁS KROÓ

Problem 1: Density of homogeneous polynomials on $\mathbf{0}$-symmetric convex bodies

Let us consider the space of **homogeneous** polynomials of degree n of d variables

$$H_n^d := \Big\{ \sum_{|\mathbf{k}|=n} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^d,\ \mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}_+^d,\ |\mathbf{k}| = k_1 + \cdots + k_d \Big\},$$

and denote by $H^d := \bigcup_n H_n^d$ the set of all homogeneous polynomials. Given a convex compact $\mathbf{0}$-symmetric set $K \subset \mathbb{R}^d, d \geq 2$ with boundary ∂K we consider the density problem for homogeneous polynomials in *uniform norm* $\|f\|_{\partial K} := \sup_{x \in \partial K} |f(x)|$ for continuous functions $f \in C(\partial K)$. (In view of homogeneity of polynomials the approximation is not possible inside the domain.) Also $h \in H_n^d$ has the same parity as n, its even for even n and odd when n is odd, thus at least two homogeneous polynomials are need for approximating f!

Conjecture 1. *For any convex compact $\mathbf{0}$-symmetric body $K \subset \mathbb{R}^d, d \geq 2$ the set $H^d + H^d$ is dense in $C(\partial K)$, that is $\forall f \in C(\partial K)$ and $\forall \epsilon > 0, \exists h \in H^d + H^d$ so that $\|f - h\|_{\partial K} \leq \epsilon$.*

The density of $H^d + H^d$ in $C(\partial K)$ has been verified in 3 cases:

(i) When $d = 2$ and K is any $\mathbf{0}$-symmetric convex body in $\mathbb{R}^2$.
(ii) For any $\mathbf{0}$-symmetric convex **polytope** in $\mathbb{R}^d, d > 2$.
(iii) For any $\mathbf{0}$-symmetric **regular** convex body $K \subset \mathbb{R}^d, d > 2$. (Regular convex body $\Leftrightarrow$ unique supporting hyperplane at any point on its boundary.)

In its full generality the conjecture is **still open**. See [1], [4], [6] for details.

Problem 2: Optimal admissible polynomial meshes

Consider now the space of polynomials of total degree at most n of d variables given by

$$P_n^d := \Big\{ \sum_{|\mathbf{k}|\leq n} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbb{R},\ \mathbf{x} \in \mathbb{R}^d,\ \mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}_+^d,\ |\mathbf{k}| = k_1 + \cdots + k_d \Big\},$$

and as above denote by $\|f\|_K := \sup_{x \in K} |f(x)|$ the uniform norm on the set $K \subset \mathbb{R}^d$, $d \geq 2$.

Definition 1. *A family of discrete sets $\mathbf{Y} = \{Y_n \subset K, n \in \mathbb{N}\}$ is called an* **optimal admissible mesh** *in $K \subset \mathbb{R}^d, d \geq 2$ if there exist positive constants c_1, c_2 depending only on K such that*

$$\|p\|_K \leq c_1 \|p\|_{Y_n}, \quad \forall p \in P_n^d,\ \forall n \in \mathbb{N},$$

where the cardinality of Y_n grows as $\mathrm{card}(Y_n) \leq c_2 n^d$, $\forall n \in \mathbb{N}$.

Problem 1. *Which sets $K \subset \mathbb{R}^d, d \geq 2$ possess optimal admissible meshes?*

Date: February 23, 2013.

The existence of optimal admissible meshes has been verified in two cases:

(i) For any convex **polytope** in $\mathbb{R}^d, d \geq 2$.

(ii) For any **star like** set $K \subset \mathbb{R}^d, d \geq 2$ with $C^{2-\frac{2}{d}}$ boundary. (The set is star like with respect to some $\mathbf{a} \in K$ if whenever $\mathbf{x} \in K$ we have that $[\mathbf{x}, \mathbf{a}] \in K$.)

Conjecture 2. *Any convex body in $\mathbb{R}^d$ possesses an optimal admissible mesh.*

See [2], [3] for details.

Problem 3: Markov type problem in the L^2 norm on the disc

Let $L_2(D)$ be the space of square integrable real functions on the disc $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \subset \mathbb{R}^2$ endowed with the norm

$$\|f\| := \left(\int_D |f|^2 d\mathbf{x}\right)^{1/2}.$$

Then the L^2 Markov problem consists in determining the norm of the differentiation operator on the space of polynomials P_n^2 of two variables, that is finding the quantity

$$M(P_n^2, D) := \sup_{p \in P_n^2} \frac{\|Dp\|}{\|p\|}$$

where $D := |\nabla p|$ and ∇p stands for the gradient vector of p.

A variational argument yields that if $p^* \in P_n^2$ is an extremal polynomial for which the above sup is attained then for every $p \in P_n^2$ we have

$$\iint_D (\Delta p^* + M(P_n^2, D)^2 p^*)p = \int_{\partial D} (-p_y^* dx + p_x^* dy)p,$$

where Δp^* stands for the Laplace operator and $\partial D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is the unit circle.

In particular the above orthogonality relation yields that $\Delta p^* + M(P_n^2, D)^2 p^*$ is orthogonal to P_{n-2}^2 with the weight $1 - x^2 - y^2$. This may lead to explicit differential equations for p^*.

Problem 2. *Based on the above orthogonality relation find p^* and $M(P_n^2, D)$.*

See [5] for details.

References

[1] D. Benko and A. Kroó, A Weierstrass-type theorem for homogeneous polynomials, *Trans. Amer. Math. Soc.* **361** (2009), 1645–1665.

[2] J.P. Calvi and N. Levenberg, Uniform approximation by discrete least squares polynomials, *J. Approx. Theory* **152** (2008), 82–100.

[3] A. Kroó, *On optimal polynomial meshes*, *J. Approx. Theory* **163** (2011), 1107–1124.

[4] A. Kroó and J. Szabados, On the density of homogeneous polynomials on regular convex surfaces, *Acta Sci. Math. (Szeged)* **75** (2009), 143–159.

[5] A. Kroó and J. Szabados, On the exact L^2 Markov inequality on some unbounded domains in $\mathbb{R}^d$, *J. Approx. Theory* **164** (2012), 391–405.

[6] P. Varjú, Approximation by homogeneous polynomials, *Constr. Approx.* **26** (2007), 317–337.

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary

E-mail address: kroo.andras@renyi.mta.hu