On para-orthogonal polynomials

Ruymán Cruz-Barroso

Departament of Mathematical Analysis, La Laguna University, Tenerife, Canary Islands, Spain.
rcruzb@ull.es

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Notation

- $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, the unit circle.
- $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the open unit disk.
- $\mathbb{P} := \mathbb{C}[z]$ space of polynomials.
- $\mathbb{P}_n := \mathrm{span}\{1, z, \dots, z^n\}$.

- For a polynomial $P_n \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ we define its reversed (or reciprocal) as

$$P_n^*(z) = z^n \overline{P_n(1/\bar{z})} \in \mathbb{P}_n.$$

$$P_n(z) = a_0 + a_1 z + \cdots + a_n z^n \quad \Rightarrow$$
$$P_n^*(z) = \overline{a_n} + \overline{a_{n-1}} z + \cdots + \overline{a_0} z^n.$$

Observe that $P_n^*(z) = z^n \overline{P_n(z)}$ when $z \in \mathbb{T}$.

Notation

- μ: a positive Borel measure on $\mathbb{T}$.
- $L_2^\mu(\mathbb{T})$: Hilbert space of measurable functions ψ for which $\int_{-\pi}^{\pi} \left|\psi(e^{i\theta})\right|^2 d\mu(\theta) < +\infty$.
- Inner product induced by μ:

$$\langle \phi, \varphi \rangle_\mu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(z)\overline{\varphi(z)} d\mu(\theta), \text{ where } \phi, \varphi \in L_2^\mu(\mathbb{T}) \text{ and } z = e^{i\theta}.$$

Notice that

$$\langle z^n f, g \rangle_\mu = \langle f, z^{-n} g \rangle_\mu.$$

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Szegő polynomials

- $\{\rho_n\}_{n=0}^{\infty}$: Szegő polynomials.

  $\rho_0 = \rho_0^* \equiv 1$, $\rho_n(z) \in \mathbb{P}_n$ monic satisfying

  $$\langle \rho_n(z), z^s \rangle_\mu = \langle \rho_n^*(z), z^t \rangle_\mu = 0 \quad \text{for all} \quad \begin{array}{l} s = 0, 1, \ldots, n-1, \\ t = 1, 2, \ldots, n, \end{array}$$

  and

  $$\langle \rho_n(z), z^n \rangle_\mu = \langle \rho_n^*(z), 1 \rangle_\mu > 0.$$

Szegő recurrence

- $\rho_n(z) \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ and $\rho_n(z) \perp \operatorname{span}\{1, z, \ldots, z^{n-1}\}$.
- $z\rho_n(z) \in \mathbb{P}_{n+1} \backslash \mathbb{P}_n$ and $z\rho_n(z) \perp \operatorname{span}\{z, \ldots, z^n\}$ (same orthogonality conditions as ρ_n^*)
- Two possibilities:
  1. If $\langle z\rho_n(z), 1\rangle_\mu = 0$, then $\rho_{n+1}(z) = z\rho_n(z)$.
  2. If $\langle z\rho_n(z), 1\rangle_\mu \neq 0$, then set $R_{n+1}(z) := z\rho_n(z) + \delta_{n+1}\rho_n^*(z)$.

     Since $R_{n+1}(z) \in \mathbb{P}_{n+1} \backslash \mathbb{P}_n$ and $R_{n+1}(z) \perp \operatorname{span}\{z, \ldots, z^n\}$, we can choose δ_{n+1} so that $\langle R_{n+1}(z), 1\rangle_\mu = 0$.

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     $\delta_{n+1} = \rho_{n+1}(0)$ is called $(n+1)$-th <u>Verblunsky coefficient</u>, $\delta_{n+1} \in \mathbb{D}$.

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Szegő recurrence

$$\rho_{n+1}(z) = z\rho_n(z) + \delta_{n+1}\rho_n^*(z).$$

Taking $*$-operation:

$$\rho_{n+1}^*(z) = \overline{\delta_{n+1}} z\rho_n(z) + \rho_n^*(z).$$

Szegő recurrence

- Szegő recurrence:

$$\begin{pmatrix} \rho_{n+1}(z) \\ \rho^*_{n+1}(z) \end{pmatrix} = \begin{pmatrix} z & \delta_{n+1} \\ \overline{\delta_{n+1}} z & 1 \end{pmatrix} \begin{pmatrix} \rho_n(z) \\ \rho^*_n(z) \end{pmatrix}, \quad n = 0, 1, \ldots.$$

Initial conditions: $\rho_0 = \rho^*_0 \equiv 1 \quad (\delta_0 = 1)$.

- From ρ_n and ρ^*_n, we can compute δ_{n+1} from

$$\delta_{n+1} = \frac{\langle z\rho_n(z), 1\rangle_\mu}{\langle \rho^*_n(z), 1\rangle_\mu},$$

and then use the recursion to compute ρ_{n+1} and ρ^*_{n+1}.

- Assumption: the family of trigonometric moments is known in advance:

$$c_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} d\mu(\theta) = \langle z^k, 1\rangle_\mu.$$

Zeros

Drawback!: the zeros of ρ_n lie on $\mathbb{D}$, $n \geq 1$.

It is of interest (e.g., for the construction of quadrature formulas on the unit circle) to have distinct zeros on $\mathbb{T}$ (like in the real axis case).

$\rightsquigarrow$ Para-orthogonality + invariance:

W.B. Jones, O. Njåstad, W. Thron.- *Moment theory, orthogonal polynomials, quadrature and continued fractions associated with the unit circle*, Bull. Lond. Math. Soc. 21 (1989) 113-152.

Definition of para-orthogonality and invariance

A sequence of polynomials $\{\Phi_n\}_{n\geq 2}$ is a family of POPUC if

1. $\deg(\Phi_n) = n$,
2. The following orthogonality conditions holds:

$$\langle \Phi_n, 1\rangle_\mu \neq 0, \quad \langle \Phi_n, z^m\rangle_\mu = 0,\ 1 \leq m \leq n-1, \quad \langle \Phi_n, z^n\rangle_\mu \neq 0.$$

Notice: the zeros of a POPUC can be everywhere on $\mathbb{C}$.

A sequence of polynomials $\{\Phi_n\}_{n\geq 2}$ is said to be invariant if there exists $\chi_n \in \mathbb{T}$ s.t.

$$\Phi_n^*(z) = \chi_n \Phi_n(z).$$

Main Theorem: characterization result

- Theorem (Jones, Njåstad, Thron; 1989):

  1. $\Phi_n \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ is para-orthogonal and invariant $\iff$ There exists $C_n \neq 0$ and $\tilde{\tau}_n \in \mathbb{T}$ such that

  $$\Phi_n(z) = C_n \cdot \left[\rho_n(z) - \overline{\tilde{\tau}_n} \rho_n^*(z)\right] = \tilde{C}_n \cdot \left[z\rho_{n-1}(z) - \overline{\tau_n} \rho_{n-1}^*(z)\right],$$

  Relation: $\tilde{C}_n = C_n \cdot \left(1 - \overline{\tilde{\tau}_n \cdot \delta_n}\right) \neq 0$ and $\tau_n = \frac{\tilde{\tau}_n - \overline{\delta_n}}{1 - \tilde{\tau}_n \delta_n} \in \mathbb{T}$.

  2. n zeros of Φ_n: distinct and located on $\mathbb{T}$.

- Remark: We can compute Φ_n from the Szegő recurrence for the data

$$\{\delta_0 = 1, \delta_1, \ldots, \delta_{n-1}, \delta_n \to -\overline{\tau_n}\}.$$

- Notation: monic family of invariant POPUC: $\Phi_n(z, \tau_n)$.

Aim of the talk

- Invariant para-orthogonal polynomials seems to play the same role as orthogonal polynomials for measures supported on the real axis.

  Invariance $\approx$ similar role as real coefficients in orthogonal polynomials on the real line.

- They are not unique, depend on a unimodular free parameter

  $\rightsquigarrow$ for a fixed measure μ and degree n, there exists a one-parameter family of invariant para-orthogonal polynomials of degree n for μ. Characterization: as before.

Aim of the talk

To discuss different roles that can play the parameter $\tau_n \in \mathbb{T}$.

First role: interlazing properties for the zeros

- Several results: interlazing properties involving $\Phi_n(z, \tau_n)$ vs $\Phi_n(z, \tilde{\tau}_n)$ and $\Phi_n(z, \tau_n)$ vs $\Phi_{n+1}(z, \tilde{\tau}_{n+1})$: L. Golinskii, C.M.V., M. Wong and B. Simon.

- **Theorem (Simon, 2007)**: Setting $\lambda_n = \overline{\tau_n \tau_{n+1}} \frac{1+\tau_n \overline{\delta_{n+1}}}{1+\overline{\tau_n}\delta_{n+1}} \in \mathbb{T}$, then one of the following two possibilities holds:

  1. Φ_n and Φ_{n+1} have no zeros in common. In that case, λ_n is not a zero of either, and
  $\{\text{zeros of } \Phi_n\} \cup \{\lambda_n\}$ strictly interlace $\{\text{zeros of } \Phi_{n+1}\}$.
  2. Φ_n and Φ_{n+1} have a single zero in common. In that case, λ_n is that zero, and
  $\{\text{zeros of } \Phi_n\}$ strictly interlace $\{\text{zeros of } \Phi_{n+1}\} \setminus \{\lambda_n\}$.

B. Simon.- *Rank one perturbations and the zeros of para-orthogonal polynomials on the unit circle*, J. Math. Anal. Appl. 329 (2007), 376-382.

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First role:

First role of the parameter τ_n:

The parameter τ_n can be chosen to get interlacing properties for the zeros of invariant para-orthogonal polynomials.

First role: the symmetric case

- If ω symmetryc: $d(\omega)(z) = d\omega(\overline{z})$, $z \in \mathbb{T}$, $\rightsquigarrow$ $\delta_n \in (-1, 1)$, $\rightsquigarrow$ the zeros of $\Phi_n(z, \pm 1)$ are real (± 1) or appear in complex conjugate pais.
- In Simon's result: $\lambda_n = \tau_n \tau_{n+1}$. Distribution:

Case	$\tau = -1$	$\tau = 1$
$n = 2m$, $m = 2k$	$\{\pm z_l, \pm \overline{z_l}\}_{l=1}^{k}$	$\{\pm 1, \pm i\} \cup \{\pm z_l, \pm \overline{z_l}\}_{l=1}^{k-1}$
$n = 2m$, $m = 2k+1$	$\{\pm i\} \cup \{\pm z_l, \pm \overline{z_l}\}_{l=1}^{k}$	$\{\pm 1\} \cup \{\pm z_l, \pm \overline{z_l}\}_{l=1}^{k}$
$n = 2m+1$	$\{-1\} \cup \{z_l, \overline{z_l}\}_{l=1}^{m}$	$\{1\} \cup \{-z_l, -\overline{z_l}\}_{l=1}^{m}$

Second role: Szegő quadrature formulas

- Aim: to estimate integrals of the form

$$I_\mu(f) := \int_{\mathbb{T}} f(z)d\mu(z)$$

by n-point quadrature rules,

$$I_n(f) = \sum_{k=1}^{n} \lambda_k f(z_k).$$

- Since every continuous function on $\mathbb{T}$ can be uniformly approximated on $\mathbb{T}$ by Laurent polynomials $\Rightarrow$ we look for exactness in spaces of Laurent polynomials. The result is not always true for ordinary polynomials.
- Notation:
  - $\Lambda := \mathrm{span}\{z^k : k \in \mathbb{Z}\}$,
  - Given $p, q \in \mathbb{Z}$, $p \leq q$: $\Lambda_{p,q} := \mathrm{span}\{z^p, \ldots, z^q\}$.
- Look for exactness in $\Lambda_{p,q}$ with $q - p$ as large as possible.

Second role: Szegő quadrature formulas

Theorem (Jones, Njåstad, Thron; 1989): $I_\mu(L) = I_n(L)$, for all $L \in \Lambda_{-(n-1),n-1}$ (Szegő quadrature formula), if and only if,

- The nodes are the zeros of $\Phi_n(z, \tau_n)$, for some $\tau_n \in \mathbb{T}$,
- The weights can be computed by

$$\lambda_j = \frac{-1}{2z_j} \cdot \frac{A_n(z_j, \tau_n)}{B_n'(z_j, \tau_n)} > 0, \quad j = 1, \ldots, n.$$

$\rightsquigarrow$ $A_n(z, \tau_n)$: n-th second kind para-orthogonal polynomial for μ (Verblunsky coefficients: $\{-\delta_n\}_{n \geq 1}$).

$\rightsquigarrow$ Moreover, $\Lambda_{-(n-1),n-1}$ is a maximal domain of validity: there cannot exist an n-point q.f. which correctly integrates every function $f \in \Lambda_{-n,n-1}$ or every function $f \in \Lambda_{-(n-1),n}$.

Second role: Szegő quadrature formulas

Main difference with respect to Gaussian rules (measures supported on $\mathbb{R}$): for each n we have a one-parameter family of Szegő rules, $2n$ parameters to be determined but exact in a subspace of Laurent polynomials of dimension $2n-1$.

R. Cruz-Barroso, L. Daruis, P. González-Vera and O. Njåstad.- *Sequences of orthogonal Laurent polynomials, bi-orthogonality and quadrature formulas on the unit circle*, JCAM 200(1) (2007) 424-440.

Second role of the parameter τ_n:

The parameters τ_n determines the set of (positive) Szegő quadrature formulas for μ.

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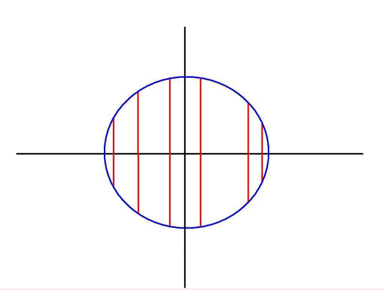
Second role: the symmetric case

- Joukowsky transform: $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $x = J(z)$. Maps $\mathbb{D}$ onto the cut Riemann sphere $\overline{\mathbb{C}} \backslash [-1,1]$ and $\mathbb{T}$ onto $[-1,1]$.
- μ symmetric measure on $\mathbb{T} \Rightarrow \sigma$ measure on $[-1,1]$, related to μ by the Joukowsky transform: $\mu(\theta) = \sigma(\cos\theta)|\sin\theta|$.



- **A. Bultheel, L. Daruis and P. González-Vera**.- *A connection between quadrature formulas on the unit circle and the interval*, JCAM 132 (2001) 1-14.

Second role: the symmetric case

- μ symmetric measure on $\mathbb{T}$, σ measure on $[-1,1]$ by Joukowsky.
- Connection between real Szegő quadrature formulas for ω and Gauss-type rules for σ:

$$\begin{aligned}
\phi_{2n}(z,1) &\rightsquigarrow && (n+1) \text{ point Gauss-Lobatto rule for } \sigma, \\
\phi_{2n}(z,-1) &\rightsquigarrow && n \text{ point Gauss rule for } \sigma, \\
\phi_{2n+1}(z,1) &\rightsquigarrow && (n+1) \text{ point Gauss-Radau(1) for } \sigma, \\
\phi_{2n+1}(z,-1) &\rightsquigarrow && (n+1) \text{ point Gauss-Radau(-1) for } \sigma.
\end{aligned}$$

- **R. Cruz-Barroso, C. Díaz Mendoza and F. Perdomo Pío**.- *A connection between Szegő-Lobatto and quasi Gauss-type quadrature formulas*, JCAM, To appear 2015.
  - Interlazing property between Gauss, Radau and Lobatto nodes (alternative proof due to A. Markov).
  - Characterization of quasi Gauss-type formulas (alternative proof due to B. Beckermann et. al., Calcolo, 2014).

Third role: Szegő q.f. exact in a subspace of dimension $2n$

- A nested sequence of subspaces of Laurent polynomials can be constructed to get a subspace of exactness of dimension $2n$.
- Idea: change the basis. Define $\sigma_n^{\pm} := \frac{\delta_n \pm \tau_n}{\overline{\delta_n \pm \tau_n}}$ and

$$\begin{pmatrix} z^k & z^{-k} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ s_k^{\pm} & -s_k^{\pm} \end{pmatrix} =: \begin{pmatrix} e_k^{\pm} & e_{-k}^{\pm} \end{pmatrix}.$$

- It is easy to check:

$$\begin{aligned} \Lambda_{-k,k} &= \operatorname{span}\left\{z^{-k}, \ldots, z^k\right\} \\ &= \operatorname{span}\left\{e_{-k}^{+}, \ldots, e_k^{+}\right\} \; . \\ &= \operatorname{span}\left\{e_{-k}^{-}, \ldots, e_k^{-}\right\} \end{aligned}$$

Third role: Szegő q.f. exact in a subspace of dimension $2n$

Theorem (J.C. Santos-León and O. Njåstad, 2007):

- The Szegő q.f. based on the zeros of $\Phi_n(z, -\tau_n)$ is exact in $\mathcal{R}_n^+ := \mathrm{span}\left\{e_l^+, \quad l = -(n-1), \ldots, n-1, n\right\}$.
- The Szegő q.f. based on the zeros of $\Phi_n(z, \tau_n)$ is exact in $\mathcal{R}_n^- := \mathrm{span}\left\{e_l^-, \quad l = -n, -(n-1), \ldots, n-1\right\}$.

Notice: $\Lambda_{-(n-1),n-1} \subset \mathcal{R}_n^\pm \subset \Lambda_{-n,n}$, and $\dim\left(\mathcal{R}_n^\pm\right) = 2n$.

Third role of the parameter τ_n:

The parameter τ_n defines de subspace of dimension 2n that in which the Szegő quadrature formula is exact.

Third role: Szegő q.f. exact in a subspace of dimension $2n$

Theorem (J.C. Santos-León and O. Njåstad, 2007):

- The Szegő q.f. based on the zeros of $\Phi_n(z, -\tau_n)$ is exact in $\mathcal{R}_n^+ := \operatorname{span}\left\{e_l^+, \;\; l = -(n-1), \ldots, n-1, n\right\}$.
- The Szegő q.f. based on the zeros of $\Phi_n(z, \tau_n)$ is exact in $\mathcal{R}_n^- := \operatorname{span}\left\{e_l^-, \;\; l = -n, -(n-1), \ldots, n-1\right\}$.

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Third role of the parameter τ_n:

The parameter τ_n defines de subspace of dimension $2n$ that in which the Szegő quadrature formula is exact.

Fourth role: Szegő-Radau rules

- The parameter τ_n can be always chosen to fix an arbitrary prefixed node in a Szegő quadrature formula $\Rightarrow$ Szegő-Radau rules are trivial!
- Completely different situation w.r.t. measures supported on the real axis.
- **A. Bultheel, R. Cruz-Barroso and M. Van-Barel**.- *On Gauss-type quadrature formulas with prescribed nodes anywhere on an interval of the real line*, Calcolo 47(1) 82010) 21-48.
- If we want $\alpha \in \mathbb{T}$ to be a node $\Rightarrow$ take $\tau_n = \alpha^{n-2}\frac{\overline{\rho_{n-1}(\alpha)}}{\rho_{n-1}(\alpha)} \in \mathbb{T}$.

Fourth role: Szegő-Radau rules

Theorem (B. Simon, 2007): if $\{\Phi_n(z,\tau_n)\}_{n\geq 2}$ have a common zero at $z=\alpha$, then the parameters τ_n are inductively given by

$$\tau_1 = \overline{\alpha}, \qquad \tau_{n+1} = \overline{\alpha}\cdot\overline{\tau_n}\cdot\left(\frac{1+\tau_n\overline{\delta_{n+1}}}{1+\overline{\tau_n}\delta_{n+1}}\right).$$

Fourth role of the parameter τ_n:

The parameter τ_n allows to fix one prescribed node in a Szegő quadrature formula and it still has maximal domain of exactness.

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Fifth role: Anti-Szegő rules

Kim and Reichel (2007), inspired by Laurie (real line situation).

- $I_n(f) = \sum_{i=1}^{n} \lambda_i f(z_i)$, a Szegő q.f. for $I_\mu(f)$. Nodal: $\phi_n(z, \tau_n)$.
- $\tilde{I}_n(f) = \sum_{i=1}^{n} \tilde{\lambda}_i f(\tilde{z}_i)$: Anti-Szegő q.f. such that

$$I_\mu(L) - I_n(L) = -c \cdot \left[I_\mu(L) - \tilde{I}_n(L) \right], \quad \forall L \in \Lambda_{-n,n}, \quad c > 0.$$

- This is equivalent to:
  - $I_\mu(L) - I_n(L) = \frac{1}{1+c} \left[\tilde{I}_n(L) - I_n(L) \right]$: the error (left hand side) can be exactly computable (right hand side), for all $L \in \Lambda_{-n,n}$. This suggests to give as an estimation of the error the expression in the right hand side, for integrands not in $\Lambda_{-n,n}$.
  - $I_\mu(L) = \frac{1}{1+c} \tilde{I}_n(L) + \frac{c}{1+c} I_n(L)$: the expression in the right hand side suggest to be a better approximation of $I_\mu(L)$ than $I_n(L)$, also for integrands not in $\Lambda_{-n,n}$.

Fifth role: Anti-Szegő rules

How to construct the Anti-Szegő rule?

- Consider the modified linear functional $\tilde{I} := (1+c)I - c \cdot I_n$.
- The Verblunsky coefficients related to $\tilde{I}$ are given in terms of the corresponding to I:

$$\tilde{\delta}_j = \delta_j, \quad j = 0, \ldots, n-1, \text{ and } \tilde{\delta}_n = (1+c)\delta_n - c \cdot \tau_n.$$

Thus,

$$\tilde{\rho}_j(z) = \rho_j(z), \quad j = 0, \ldots, n-1, \text{ and } \tilde{\rho}_n(z) = z\rho_{n-1}(z) + \tilde{\delta}_n \rho^*_{n-1}(z).$$

- We need to force then $\tilde{\delta}_n \in \mathbb{T}$.

Fifth role: Anti-Szegő rules

Theorem: for all $\tau \in \mathbb{T}$, there exists a unique $c > 0$ given by

$$c = \frac{1 - |\delta_n|^2}{1 + |\delta_n|^2 - 2\mathrm{Re}\,(\delta_n \overline{\tau_n})}$$

such that $\tilde{\delta}_n \in \mathbb{T}$.

Fifth role of the parameter τ_n:

The parameter $\tau_n = \tilde{\delta}_n \in \mathbb{T}$ defines an Anti-Szegő quadrature formula with nodes the zeros of $\phi_n(z, \tau_n)$ which is useful to estimate the error of I_n or find a more accurate quadrature formula.

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Sixth role: connection with reproducing kernels

Reproducing kernel: $\mathcal{K}_n(z,\xi) = \sum_{k=0}^{n} \varphi_k(z)\overline{\varphi_k(\xi)}$, $\quad z, \xi \in \mathbb{C}$,
reproducing property: $\langle p(z), \mathcal{K}_n(z,\xi)\rangle_\omega = p(\xi)$, $\quad \forall p \in \mathbb{P}_n, \quad \forall \xi \in \mathbb{C}$.
$\varphi_k(z)$: k-th orthonormal Szegő polynomial.

- Properties of $\mathcal{K}_n(z,\xi)(1 - z\overline{\xi}) \in \mathbb{P}_{n+1} \backslash \mathbb{P}_n$ (in variable z):
  - ⇝ For $\xi \in \mathbb{C}\backslash\{0\}$, it is para-orthogonal with respect to ω.
  - ⇝ If $\xi \in \mathbb{T}$, then it is invariant.
  - ⇝ If $\xi \in \mathbb{T}$: it has $n+1$ distinct zeros on $\mathbb{T}$, included ξ.
  - ⇝ If $\xi \in \mathbb{T}$: $\mathcal{K}_n(z,\xi)$ is invariant and para-orthogonal with respect to $d\mu(\theta) = |\xi - e^{i\theta}|^2 d\omega(\theta)$.
  - ⇝ **Christoffel-Darboux**

$$\mathcal{K}_n(z,\xi) = \frac{\varphi^*_{n+1}(z)\overline{\varphi^*_{n+1}(\xi)} - \varphi_{n+1}(z)\overline{\varphi_{n+1}(\xi)}}{1 - z\overline{\xi}}.$$

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Sixth role: connection with reproducing kernels

Sixth role of the parameter τ_n:

The election of the parameter

$$\tau_{n+1} = \xi^{n+1} \frac{\overline{\rho_{n+1}(\xi)}}{\rho_{n+1}(\xi)}$$

let us link $\mathcal{K}_n(z,\xi)(1-z\overline{\xi})$ *with* $\Phi_{n+1}(z,\tau_{n+1})$.

Seventh (last!) role: three-term para-orthogonal recursion

Set τ_1, τ_2 arbitrary $\rightsquigarrow$ construct $\{\tau_n\}_{n\geq 1}$ recursively by

$$\tau_{n+1} = \frac{\tau_n - \delta_{n-1}}{\overline{\tau_n} - \overline{\delta_{n-1}}} \cdot \frac{\overline{\tau_{n-1}} - \overline{\delta_{n-2}}}{\tau_{n-1} - \delta_{n-2}},$$

Theorem (K. Castillo, R. Cruz-Barroso and F. Perdomo-Pío, submitted 2015): A three-term recurrence relation for invariant para-orthogonal polynomials of the form

$$\Phi_{n+1}(z, \tau_{n+1}) = (z + \beta_n)\Phi_n(z, \tau_n) - \gamma_n z \Phi_{n-1}(z, \tau_{n-1}),$$

where

$$\beta_n = \frac{\tau_n}{\tau_{n+1}} \in \mathbb{T}, \quad \gamma_n = \frac{\tau_n - \delta_{n-1}}{\tau_{n-1} - \delta_{n-2}}(1 - |\delta_{n-2}|^2)\frac{\tau_{n-1}}{\tau_{n+1}}$$

holds, if and only if, the parameters τ_n's are recursively given as before.

Seventh (last!) role: three-term para-orthogonal recursion

In this particular situation:

- A Favard type theorem (spectral theorem) can be proved (converse result of the three-term recurrence).
- A Geronimus-Wendroff theorem can also be stated: given two monic polynomials ψ_n and ψ_{n+1} whose zeros are simple and strictly interlace on $\mathbb{T}$, then there exist a sequence of invariant para-orthogonal polynomials such that the above polynomials belong to it.

Seventh role of the parameter τ_n:

A particular election of the parameters τ_n's makes possible to deduce a three-term recurrence for invariant POPUC, and hence similar results as in the theory of orthogonal polynomials on the real line theory can be obtained for them.

Thanks for your attention!