A generalization of the concept of coherent pairs for discrete orthogonal polynomials and some of their applications

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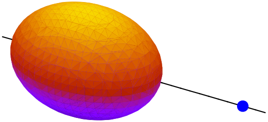
Plan

1. Introduction
2. Preliminars
3. Main results
4. An Application
5. Conclusions

178X

$$\frac{(1-\rho)\cos\gamma}{(1-2\rho\cos\gamma+\rho^2)^{\frac{3}{2}}} =$$



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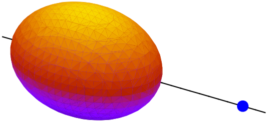

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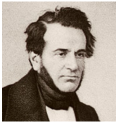
$$\int_0^1 P_n(x)P_m(x)dx = \frac{2\delta_{n,m}}{2n+1}, \qquad (1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0,$$

1826 : $\quad P_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}$



Back in time ...

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$$P_n(x) = \frac{P_{n+1}'(x)}{n+1} + f_n P_n'(x) + g_n P_{n-1}'(x)$$

On characterizations: Marcellán & Petronilho ITSF 1994, RAN JCAM 2006

Time for a new definition: coherent pair of measures

Let μ_0 and μ_1 be two "nice" measures and define the following Sobolev-inner product

$$\langle p(x), q(x)\rangle_\lambda = \int_{\mathbb{R}} p(x)q(x)d\mu_0 + \lambda \int_{\mathbb{R}} p^{(m)}(x)q^{(m)}(x)d\mu_1, \lambda > 0, m \in \mathbb{N}$$

and let $\{S_n(x;\lambda)\}_{n\geq 0}$ be the sequence of monic Sobolev pol. orthogonal w.r.t. the above inner product.

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There are a huge amount of interesting results concerning the algebraic and analytic properties of $S_n(x;\lambda)$ but, in general these results are quite complicated to obtain for an arbitrary pair (μ_0, μ_1).

Among all the possible approaches, there is one that have been used extensively in the last years: the *coherent pair of measures* (or functionals) and it was introduced by Iserles et al. in JAT 1991.

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The idea is: If $\{R_n(x)\}_{n\geq 0}$ is the OPS $\perp$ to μ_0 and $\{P_n(x)\}_{n\geq 0}$ is $\perp$ to μ_1, force that between R_n and P_n exists some kind of linear **structure** relation.

Definition (Iserles et al. 1991)

We said that pair (μ_0, μ_1) is a coherent pair of positive Borel measures if their corresponding SMOPs $\{R_n(x)\}_{n\geq 0}$ and $\{P_n(x)\}_{n\geq 0}$ satisfy the structural relation

$$R_n(x) = \frac{1}{n+1} P'_{n+1}(x) + \alpha_n P'_n(x), \quad n \geq 0. \tag{1}$$

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One question that immediately arises is: If (1) holds there is any relation between the measures? and viceversa: for which kind of measures (1) holds?

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If $\alpha_n \neq 0$ then it is well known (Marcellán & Petronilho 1995, Meijer 1997) that (1) holds iff one the measures is a classical one.

An extension of the coherent pair notion was given by Martínez-Finkelshtein in JCAM 1998:

$$R_n(x) = \frac{1}{n+1} P'_{n+1}(x) + \sum_{j=n-k}^{n} \alpha_{n,j}^{n} P'_j(x), \quad \alpha_{n,n-k} \neq 0$$

In this case the pair (μ_0, μ_1) is said to be a k-coherent pair.

Some non trivial examples of 1, 2 and 3 coherent pairs were encountered by Marcellán, Martínez-Finkelshtein, and Moreno-Balcazar in 2001 in Chicho's book.

- Petronilho JMAA 2006: there is any relation between the lin. fun. $\mathcal{U}$ and $\mathcal{V}$ if

$$\sum_{i=0}^{M} a_{i,n} P_{n-i}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n-i}(x), \quad n \geq 0 \quad P_n \perp \mathcal{U} \quad \& \quad Q_n \perp \mathcal{V}$$

Going back to our problem: linear relations between OPS

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- de Jesus & Petronilho JMAA 2008, JCAM 2013: the general case with a partial resolution

$$\sum_{i=0}^{M} a_{i,n} D^m P_{n+m-i}(x) = \sum_{i=0}^{N} b_{i,n} D^k Q_{n+k-i}(x), \quad n \geq 0,$$

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- de Jesus, Marcellán, Petronilho, & Pinzón-Cortés JCAM 2014: the complete solution

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and it was related with the so-called coherent pairs of measures.

A next step on this subject was done by Petronilho and de Jesus in 2008. They were interested on the study of the OPSs $(P_n)_n$ and $(R_n)_n$ orthogonal w.r.t two regular functionals $(\mathcal{U}, \mathcal{V})$, respectively s.t. the following structure relation holds

$$\sum_{i=0}^{M} a_{i,n} P_{n+m-i}^{(m)}(x) = \sum_{i=0}^{N} b_{i,n} R_{n+k-i}^{(k)}(x), \quad n \geq 0,$$

where $M, N, m, k \in \mathbb{N} \cup \{0\}$, $a_{M,n} \neq 0$ for $n \geq M$, $b_{N,n} \neq 0$ for $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ for $i > n$, or, for an easy understanding

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$$P^{(m)}_{n+m} + a_{1,n} P^{(m)}_{n+m-1} + \cdots + a_{M,n} P^{(m)}_{n+m-M} = b_{0,n} R^{(k)}_{n+k} + b_{1,n} R^{(k)}_{n+k-1} + \cdots + b_{N,n} R^{(k)}_{n+k-N}$$

Coherent pair of measures: Going further

$$P_{n+m}^{(m)} + + \cdots + a_{M,n} P_{n+m-M}^{(m)} \quad = \quad b_{0,n} R_{n+k}^{(k)} + \cdots + b_{N,n} R_{n+k-N}^{(k)}$$

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In paper with Marcellán and Pinzón-Cortéz (JCAM 2014) they **named the above pair $(\mathcal{U}, \mathcal{V})$ a (M, N)-coherence pair of functional of order (m, k)**.

$(\mathcal{U}, \mathcal{V})$ is a (M, N)-coherence pair of functional of order (m, k) if

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Remark: Notice that

1. The $(1, 0)$-coherent pair of order $(1, 0)$ is the coherent pair introduced in JAT 1991
$$R_n(x) = \frac{1}{n+1} P'_{n+1}(x) + \alpha_n P'_n(x).$$
and characterized by Meijer in 1997.

2. The $(k + 1, 0)$-coherent pair of order $(1, 0)$ is the k-coherent pair introduced by Andrei M-K in 1998.
$$R_n(x) = \frac{1}{n+1} P'_{n+1}(x) + a_{1,n} P'_n + \cdots + a_{k,n} P'_{n-k}$$

Going back for a while ...

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(M, N)-coherence pair of functional of order (m, k) is a *very natural extension*.

$$P_{n+m}^{(m)}+a_{1,n}P_{n+m-1}^{(m)}+\cdots+a_{M,n}P_{n+m-M}^{(m)}=b_{0,n}R_{n+k}^{(k)}+b_{1,n}R_{n+k-1}^{(k)}+\cdots+b_{N,n}R_{n+k-N}^{(k)}$$

In dJMPP-C JCAM 2014 a complete characterization of the (M,N)-coherent pairs of order (m,k) in terms of the semiclassical functionals has been done.

But what are these semiclassical functionals?

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Definition

A regular linear functional $\mathcal{U}$ is called *D-semiclassical* linear functional if it is regular and $\exists \sigma, \tau \in \mathbb{P}$, with $\deg(\tau(x)) \geq 1$, such that

$$\boldsymbol{D(\sigma(x)\mathcal{U}) = \tau(x)\mathcal{U}}$$

The *class* of $\mathcal{U}$ is $s := \min\max\{\deg\sigma - 2, \deg\tau - 1\} \in \mathbb{N}\cup\{0\}$, where the minimum is taken among all pairs of polynomials (σ, τ), with $\deg(\tau(x)) \geq 1$.

When $s = 0$, $\mathcal{U}$ is called a *D-classical* functional.

The corresponding SMOP $\perp \mathcal{U}$ is said to be *D*-semiclassical of class s.

The complete solution: dJMPP-C JCAM 2014

If $(\mathcal{U},\mathcal{V})$ is a (M,N)-D_ν-coherent pair of order (m,k) with $m \geq k$, then under certain conditions, $\exists \phi_{M+k+n}(x;\nu), \psi_{N+m+n}(x;\nu) \in \mathbb{P}$ such that

$$D^{m-k}[\phi_{M+k+n}(x;\nu)\mathcal{V}] = \psi_{N+m+n}(x;\nu)\mathcal{U}, \quad n \geq 0,$$

and there exist polynomials $\varphi(x;\nu)$ and $\rho(x;\nu)$ such that

$$\varphi(x;\nu)\mathcal{U} = \rho(x;\nu)\mathcal{V}.$$

Furthermore

1. If $k = m$ then $\mathcal{U}$ is a D-semiclassical linear functional iff so is $\mathcal{V}$.
2. If $m > k$, then $\mathcal{U}$ and $\mathcal{V}$ are both D-semiclassical linear functionals.

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1. If $k = m$ then $\mathcal{U}$ is a D-semiclassical linear functional iff so is $\mathcal{V}$.
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An obvious question arises: What happens in the discrete case? and in the q-case?

By $\mathbb{P}$ we will denote the linear space of polynomials with complex coeff.

We will work with the following two linear difference operators (instead of $D = \frac{d}{dx}$)

$$D_\omega : \mathbb{P} \mapsto \mathbb{P}, \quad D_\omega p(x) = \frac{p(x+\omega) - p(x)}{\omega}, \quad \omega \in \mathbb{C} \setminus \{0\},$$
$$D_q : \mathbb{P} \mapsto \mathbb{P}, \quad D_q p(x) = \frac{p(qx) - p(x)}{(q-1)x}, \qquad q \in \mathbb{C} \setminus \{0, \pm 1\}.$$

Notice that $D_1 = \Delta$ and $D_{-1} = \nabla$ are the forward and backward difference operators, respectively, and D_q is the classical q-derivative operator.

Remark: Notice that when $q \to 1$ and $\omega \to 0$ we recover the standard derivative operator.

Given a linear functional $\mathcal{U}$ a sequence or polynomials $\{P_n(x)\}_{n\geq 0}$ is called the SMOP w.r.t. $\mathcal{U}$ if

$$\deg(P_n(x)) = n \text{ and } \langle \mathcal{U}, P_n(x)P_m(x)\rangle = \xi_n \delta_{n,m},\ \xi_n \neq 0,\ n, m \geq 0.$$

In this case, $\mathcal{U}$ is said to be **regular**, and $\Upsilon_n = \det\left([u_{i+j}]_{i,j=0}^{n}\right) \neq 0,\ \forall n \geq 0.$

When $\Upsilon_n > 0,\ n \geq 0$, $\mathcal{U}$ is called *positive definite*.

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For the sake of simplicity we will use the following notation

$$P_n^{[m,\nu]}(x) := \frac{D_\nu^m P_{n+m}(x)}{\eta_{n,m,\nu}}, \quad \text{with} \quad \eta_{n,m,\omega} := (n+1)_m, \quad \eta_{n,m,q} := \frac{(q^{n+1};q)_m}{(1-q)^m}.$$

This means that the sequence of derivatives $\{P_n^{[m,\nu]}\}_n$ is always a MPS.

For each D_ω and D_q we have SODE, Rodrigues Formula, Sonin-Hahn property, Al-Salam & Chihara, MBP, etc

García, Marcellán, Salto JCAM 1995, Medem, RAN, Marcellán JCAM 2001, RAN JCAM 2006

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In particular:

$$P_n(x) = d_n D_\nu P_{n+1}(x) + f_n D_\nu P_n(x) + g_n D_\nu P_{n-1}(x)$$

where D_ν will be D_ω or D_q & D_{ν^*} will be $D_{-\omega}$ or $D_{1/q}$.

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But then ... **The history repeats**

The concept of coherent pair was extended to the OPS of a discrete variable by Area, Godoy, and Marcellán in several papers ITSF 2000, 2003, and AMC 2002. The definition is as follows

Definition

The pair (μ_0, μ_1) is a coherent pair of discrete measures supported on the linear uniform lattice $\{0, 1, 2, \cdots\}$ or the q-linear lattice $\{1, q, q^2, \cdots\}$ if the sequences $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$P_n(x) = \beta_n D_\nu R_{n+1}(x) + \alpha_n D_\nu R_n(x), \quad n \geq 0, \quad \beta_n \neq 0$$

being D_ν $(\nu = 1)$ the forward difference operator Δ or the q-Jackson derivative operator D_q as before

$$\Delta p(x) = p(x+1) - p(x), \qquad D_q p(x) = \frac{p(qx) - p(x)}{(q-1)x}, \quad p \in \mathbb{P},$$

respectively.

More recently, Marcellán and Pinzón-Cortéz Num.Alg. 2012 considered the case of $(1,1)$-coherence for the q-Jackson derivative, i.e., when the sequences $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$P_n(x) + \gamma_n P_{n-1}(x) = \beta_n D_q R_{n+1}(x) + \alpha_n D_q R_n(x), \quad n \geq 0, \quad \beta_n \neq 0$$

We are interested not only in generalizing the concept of coherence in the same fashion as it was done for the "continuous" for measures (or functionals) supported on the above discrete set of points but also in characterizing the corresponding "discrete" functionals $(\mathcal{U}, \mathcal{V})$.

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Our **aim** in this talk is to show that the technique and results of JP in JCAM 2013 and JMPP in JCAM 2014 can be "easily" extended to the discrete case.

For *obvious reasons* we will only give here some hints.

Definition

Semiclassical functional: A functional $\mathcal{U}$ is called *D_ν-semiclassical* linear functional (recall that $\nu = \omega$ or $\nu = q$) if it is regular and $\exists \sigma, \tau \in \mathbb{P}$, with $\deg(\tau(x)) \geq 1$, s.t.

$$D_\nu(\sigma(x)\mathcal{U}) = \tau(x)\mathcal{U}.$$

The *class* of $\mathcal{U}$ is $s := \min \max \{\deg \sigma - 2, \deg \tau - 1\} \in \mathbb{N} \cup \{0\}$, where the minimal is taken among all pairs of polynomials (σ, τ), with $\deg(\tau(x)) \geq 1$.

When $s = 0$, $\mathcal{U}$ is called a *D_ν-classical* functional.
When $s > 0$ the SMOP is said to be D_ν-semiclassical of class s.

Discrete semiclassical functionals

Definition

Semiclassical functional: A functional $\mathcal{U}$ is called *D_ν-semiclassical* linear functional (recall that $\nu = \omega$ or $\nu = q$) if it is regular and $\exists \sigma, \tau \in \mathbb{P}$, with $\deg(\tau(x)) \geq 1$, s.t.

$$D_\nu(\sigma(x)\mathcal{U}) = \tau(x)\mathcal{U}.$$

The *class* of $\mathcal{U}$ is $s := \min\max\{\deg\sigma - 2, \deg\tau - 1\} \in \mathbb{N} \cup \{0\}$, where the minimal is taken among all pairs of polynomials (σ, τ), with $\deg(\tau(x)) \geq 1$.

When $s = 0$, $\mathcal{U}$ is called a *D_ν-classical* functional.
When $s > 0$ the SMOP is said to be D_ν-semiclassical of class s.

Proposition (Structural relation that characterize D_ν-semiclassical OP)

Let $\{P_n(x)\}_{n\geq 0}$ be a SMOP w.r.t. a linear functional $\mathcal{U}$ and let $\sigma(x)$ be a monic polynomial. $\mathcal{U}$ is a semiclassical functional iff $\exists$ an integer $s \geq 0$ s.t.

$$\sigma(x)P_n^{[1,\nu^*]}(x) = \sum_{j=n-s}^{n+\deg(\sigma(x))} \lambda_{j,n}P_j(x), \;\; n \geq s, \quad \text{and} \quad \lambda_{n-s,n} \neq 0, \;\; n \geq s+1.$$

Proposition (Equivalence of the Pearson Equations)

The following equivalences hold

$$D_\omega\left[\sigma(x)\mathcal{U}\right] = \tau(x)\mathcal{U} \quad \Longleftrightarrow \quad D_{-\omega}\Big(\big[\sigma(x)+\omega\tau(x)\big]\mathcal{U}\Big) = \tau(x)\mathcal{U},$$

$$D_q\left[\sigma(x)\mathcal{U}\right] = \tau(x)\mathcal{U} \quad \Longleftrightarrow \quad D_{q^{-1}}\Big(\big[q\sigma(x)+(q-1)x\tau(x)\big]\mathcal{U}\Big) = \tau(x)\mathcal{U}.$$

Thus, $\mathcal{U}$ is D_ν-semiclassical iff it is D_{ν^*}-semiclassical.

Two general statements

Proposition (Equivalence of the Pearson Equations)

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Proposition (Rational modifications and semiclassical character)

If the regular linear functionals $\mathcal{U}, \mathcal{V}$ are related by

$$p(x)\mathcal{U} = r(x)\mathcal{V}, \qquad p, r \in \mathbb{P} \setminus \{0\},$$

then, $\mathcal{U}$ is D_ν-semiclassical (respectively D_{ν^*}-semiclassical) iff $\mathcal{V}$ also is D_ν-semiclassical (respectively D_{ν^*}-semiclassical). Moreover, if the class of $\mathcal{U}$ is s, then the class of $\mathcal{V}$ is at most $s + \deg(p(x)) + \deg(r(x))$.

Definition

A pair of regular linear functionals $(\mathcal{U}, \mathcal{V})$ is said to be a *(M, N)-D_ν-coherent pair of order* (m, k), with fixed $M, N, m, k \in \mathbb{N} \cup \{0\}$, if their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$P_n^{[m,\nu]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m,\nu]}(x) = Q_n^{[k,\nu]}(x) + \sum_{i=1}^{N} b_{i,n} Q_{n-i}^{[k,\nu]}(x), \quad n \geq 0,$$

where $a_{i,n}, b_{i,n} \in \mathbb{C}$, $a_{M,n} \neq 0$ for $n \geq M$, $b_{N,n} \neq 0$ for $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ if $i > n$. In addition, $(\mathcal{U}, \mathcal{V})$ is said to be a *(M, N)-D_ν-coherent pair of order m* if it is a (M, N)-D_ν-coherent pair of order $(m, 0)$.

Examples: The already mentioned discrete coherence and q- coherence pairs defined by Area, Godoy and Marcellán is a $\Delta - (1, 0)$ and $q - (1, 0)$ coherent pairs of order $(1, 0)$ and the $(1, 1) - D_q$-coherent pair defined by Marcellán and Pinzón-Cortés is a $D_q - (1, 1)$ coherent pair of order $(1, 0)$.

Theorem

Let $(\mathcal{U},\mathcal{V})$ be a (M,N)-D_ν-coherent pair of order (m,k) given by (**??**) *with $m \geq k$. Let $\mathcal{L}_{M+N} = [l_{i,j}]_{i,j=0}^{M+N-1}$ be the following matrix of order $M+N$*

$$l_{i,j} = \begin{cases} a_{j-i,j} & \text{if} \quad 0 \leq i \leq N-1 \quad \text{and} \quad i \leq j \leq M+i, \\ b_{j-i+N,j} & \text{if} \quad N \leq i \leq M+N-1 \quad \text{and} \quad i-N \leq j \leq i, \\ 0 & \text{otherwise,} \end{cases}$$

with $a_{0,j_1} = b_{0,j_2} = 1$, $0 \leq j_1 \leq N-1$, $0 \leq j_2 \leq M-1$. If $\det(\mathcal{L}_{M+N}) \neq 0$, then there exist polynomials $\phi_{M+k+n}(x;\nu)$ and $\psi_{N+m+n}(x;\nu)$, of degrees $M+k+n$ and $N+m+n$, respectively, such that

$$D_{\nu^*}^{m-k}[\phi_{M+k+n}(x;\nu)\mathcal{V}] = \psi_{N+m+n}(x;\nu)\mathcal{U}, \quad n \geq 0, \tag{2}$$

and there exist polynomials $\varphi(x;\nu)$ and $\rho(x;\nu)$ such that

$$\varphi(x;\nu)\mathcal{U} = \rho(x;\nu)\mathcal{V}. \tag{3}$$

What does this theorem means?

Under certain conditions the fact that the sequences $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfies the structural relation defined by the coherent pair notion leads to a very special relation between the functionals!

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Theorem

Furthermore

1. *If $k = m$ then $\mathcal{U}$ is a D_ν-semiclassical linear functional iff so is $\mathcal{V}$.*
2. *If $m > k$, then $\mathcal{U}$ and $\mathcal{V}$ are both D_ν-semiclassical linear functionals.*

The above Theorem gives a complete description of the D_ν-semiclassical discrete orthogonal polynomials in the framework of (M,N)-D_ν-coherence of order (m,k).

Remark: Notice that the (M,N)-D_ν-coherence of order $(0,0)$ reduces to the case of the linear relation

$$P_n(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}(x) = Q_n(x) + \sum_{i=1}^{N} b_{i,n} Q_{n-i}(x), \quad n \geq 0.$$

For this case, assuming that $\mathcal{U}$ and $\mathcal{V}$ are regular functionals Petronilho JMAA 2006 proved that they are connected by a rational rational modification $\varphi(x;\nu)\mathcal{U} = \rho(x;\nu)\mathcal{V}$.

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A more complicated problem was stated and solved for several special cases by Alfaro, Marcellán, Peña, Petronilho and Rezola in a series of papers in JMAA 2002, 2004 & 2013 and JCAM 2010. There they considered the case when **only one** of the functionals is regular.

Before showing an application let us discuss the special case when $m = k + 1$.

Theorem

Let $\mathcal{U}$ and $\mathcal{V}$ be two D_ν-semiclassical linear functionals related by a rational factor, i.e., $\exists$ monic pol. $\sigma(x)$ and $\varphi(x)$, and nonzero pol. $\tau(x)$ and $\rho(x)$, s.t.

$$D_{\nu^*}\left[\sigma(x)\mathcal{V}\right] = \tau(x)\mathcal{V}, \quad \text{and} \quad \varphi(x)\mathcal{U} = \rho(x)\mathcal{V}$$

$$\deg(\sigma(x)) = \ell, \quad \deg(\tau(x)) = t \geq 1, \quad \deg(\varphi(x)) = \jmath, \quad \deg(\rho(x)) = r,$$

hold, and let $\{P_n(x)\}_{n\geq 0} \perp \mathcal{U}$ and $\{R_n(x)\}_{n\geq 0} \perp \mathcal{V}$. Then,

$$\sum_{i=n-r-\ell}^{n+\jmath+\ell} a_{i,n} P_i^{[1,\nu]}(x) = \sum_{i=n-\jmath-s}^{n+\jmath+\ell} b_{i,n} R_i(x)$$

where $a_{n+\jmath+\ell,n} b_{n+\jmath+\ell,n} \neq 0$, for $n \geq 0$, and $s = \max\{\ell-2, t-1\}$.
Therefore, $(\mathcal{U}, \mathcal{V})$ is a $(\jmath + 2\ell + r, 2\jmath + \ell + s)$-$D_\nu$-coherent pair of order 1.

This theorem is the inverse of the previous one for the case $k = m + 1$.

This means that in the case $m = k + 1$ if $(P_n)_n$ and $(R_n)_n$ are related by the structural relation

$$P_n^{[m,\nu]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m,\nu]}(x) = Q_n^{[k,\nu]}(x) + \sum_{i=1}^{N} b_{i,n} Q_{n-i}^{[k,\nu]}(x), \quad n \geq 0,$$

then $\mathcal{U}$ and $\mathcal{V}$ should be semiclassical functionals and they should be related to each other by a rational modification.

To conclude this talk lest us consider an application of the notion of (M, N)-D_ν-coherence of order (m, k).

Let $\mathbb{P}$ be the linear space of polynomials with real coefficients and consider the Sobolev-type inner product, for fixed $m \geq 1$,

$$\langle p(x), r(x) \rangle_{\lambda,\nu} = \langle \mathcal{U}, p(x) r(x) \rangle + \lambda \langle \mathcal{V}, (D_\nu^m p)(x)(D_\nu^m r)(x) \rangle, \quad \lambda > 0,$$

where $\mathcal{U}$ and $\mathcal{V}$ are regular linear functionals.

Let $\{P_n(x)\}_{n\geq 0}$, $\{R_n(x)\}_{n\geq 0}$ and $\{S_n(x;\lambda,\nu)\}_{n\geq 0}$ be the SMOP w.r.t. $\mathcal{U}$, $\mathcal{V}$ and $\langle \cdot\,,\cdot \rangle_{\lambda,\nu}$, respectively.

We are interesting in finding the Sobolev-discrete pol. $\{S_n(x;\lambda,\nu)\}_{n\geq 0}$ orthogonal w.r.t. $\langle \cdot\,,\cdot \rangle_{\lambda,\nu}$.

The concept of coherent pairs plays a fundamental role. We assume that

$$P_n^{[m,\nu]}(x) + \sum_{i=1}^{M} a_{i,n} P_{n-i}^{[m,\nu]}(x) = R_n(x) + \sum_{i=1}^{N} b_{i,n} R_{n-i}(x), \quad n \geq 0,$$

where $a_{M,n} \neq 0$ if $n \geq M$, $b_{N,n} \neq 0$ if $n \geq N$, and $a_{i,n} = b_{i,n} = 0$ when $i > n$.

Theorem ($S_n(x;\lambda,\nu)$ in terms of P_n and R_n)

Let $(\mathcal{U},\mathcal{V})$ be a (M,N)-D_ν-coherent pair of order m given by (27), and $K=\max\{M,N\}$. Then, $S_n(x;\lambda,\nu)=P_n(x)$ for $n<m$ and

$$P_{n+m}(x)+\sum_{i=1}^{M}\frac{\eta_{n,m,\nu}a_{i,n}}{\eta_{n-i,m,\nu}}P_{n-i+m}(x)=S_{n+m}(x;\lambda,\nu)+\sum_{j=1}^{K}c_{j,n,\lambda,\nu}S_{n-j+m}(x;\lambda,\nu),$$

for $n\geq 0$, **where $\mathbf{c_{j,n,\lambda,\nu}=0}$** *for $n<j\leq K$, and, for $1\leq j\leq K$,*

$$c_{j,n,\lambda,\nu}=\frac{\eta_{n,m,\nu}}{\langle S_{n-j+m}(x;\lambda,\nu),S_{n-j+m}(x;\lambda,\nu)\rangle_{\lambda,\nu}}\left[\sum_{i=j}^{M}\frac{a_{i,n}}{\eta_{n-i,m,\nu}}\right.$$

$$\left.\langle\mathcal{U},P_{n-i+m}(x)S_{n-j+m}(x;\lambda,\nu)\rangle+\lambda\sum_{i=j}^{N}b_{i,n}\langle\mathcal{V},R_{n-i}(x)D_\nu^m[S_{n-j+m}(x;\lambda,\nu)]\rangle\right].$$

Theorem (Difference eq. for the $c_{j,k,\lambda\nu}$ and the norms of Sobolev pol.)

Let $(\mathcal{U},\mathcal{V})$ be a (M,N)-D_ν-coherent pair of order m, $K=\max\{M,N\}$, and for $n\geq 0$,

$$s_{n,\nu}=\langle S_n(x;\lambda,\nu),S_n(x;\lambda,\nu)\rangle_{\lambda,\nu},\quad \widetilde{a}_{i,n}=\frac{\eta_{n,m,\nu}}{\eta_{n-i,m,\nu}}a_{i,n},\quad \widetilde{b}_{i,n}=\eta_{n,m,\nu}b_{i,n},$$

with $a_{i,n}=b_{i,n}=0$ if $i>n$, and, $a_{0,n}=b_{0,n}=1$ for $n\geq 0$. Then

$$s_{n+m,\nu}c_{j,n+j,\lambda,\nu}=\zeta_{j,n,\lambda,\nu}-\sum_{\ell=1}^{K-j}c_{\ell,n,\lambda,\nu}c_{j+\ell,n+j,\lambda,\nu}s_{n-\ell+m,\nu},\ 0\leq j\leq K,\ n\geq 0,$$

with $s_{n,\nu}=\langle\mathcal{U},P_n^2(x)\rangle$ for $n<m$, $c_{0,n,\lambda,\nu}=1$ for $n\geq 0$, $c_{j,n,\lambda,\nu}=0$ for $n<j\leq K$, and for $0\leq j\leq K$,

$$\zeta_{j,n,\lambda,\nu}=\sum_{i=j}^{M}\widetilde{a}_{i,n+j}\widetilde{a}_{i-j,n}\langle\mathcal{U},P_{n+j-i+m}^2(x)\rangle+\lambda\sum_{i=j}^{N}\widetilde{b}_{i,n+j}\widetilde{b}_{i-j,n}\langle\mathcal{V},R_{n+j-i}^2(x)\rangle.$$

The last two theorems allow

1. recursively compute the D_ν-Sobolev SMOP $\{S_n(x;\lambda,\nu)\}_{n\geq 0}$ and the coefficients $\{c_{j,n,\lambda,\nu}\}_{n\geq 0}$, $1 \leq j \leq K$.
2. to define a recursive equation for computing the sequences $\{c_{j,n,\lambda,\nu}\}_{n\geq 0}$, $1 \leq j \leq K$, and the squared norms $\{\langle S_n(x;\lambda,\nu), S_n(x;\lambda,\nu)\rangle_{\lambda,\nu}\}_{n\geq 0}$, and thus to compute the D_ν-Sobolev SMOP $\{S_n(x;\lambda,\nu)\}_{n\geq 0}$.

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Remark: Taking the limit $q \to 1$ and $\omega \to 0$ we recover the results by Jesus & Petronilho JMAA 2008, JCAM 2013 and Jesus, Marcellan, Petronilho & Pinzón-Cortés JCAM 2014.

Interesting application in the continuous case

Let I be a open interval of the real line and let $W^{m,2}[I,\mu_0,\mu_1]$ be the Sobolev space of smooth functions

$$W^{m,2}[I,\mu_0,\mu_1] = \left\{ f : I \to \mathbb{R} \,|\, f \in L^2_{\mu_0}(I)\,,\; f^{(m)} \in L^2_{\mu_1}(I) \right\}.$$

For $f \in W^{m,2}[I,\mu_0,\mu_1]$ we have the Fourier-Sobolev series with respect to the Sobolev SMOP $\{S_n(x;\lambda)\}_{n\geq 0}$,

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f_n}{s_n} S_n(x;\lambda),$$

where

$$f_n \equiv f_n(\lambda) := \langle f(x), S_n(x;\lambda)\rangle_\lambda, \quad \text{and} \quad s_n \equiv s_n(\lambda) := \|S_n\|^2_\lambda, \quad n \geq 0.$$

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In Jesus, Marcellan, Petronilho & Pinzón-Cortés JCAM 2014 an algorithm was developed to compute the Fourier-Sobolev series with a very nice accuracy in the case when the involved measures are **coherent measures**.

Let $d\mu^{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}\chi_{(-1,1)}(x)dx$, $\alpha, \beta > -1$ and let $\{\widehat{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ be its corresponding SMOP.

Let $d\mu^{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta\chi_{(-1,1)}(x)dx$, $\alpha,\beta>-1$ and let $\{\widehat{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$ be its corresponding SMOP.

Thus, the measures $d\mu_0 := d\mu^{\alpha-3,\beta-4}$ and $d\mu_1 := d\mu^{\alpha-2,\beta}$ form a $(2,1)$-**coherent pair of order** 3, with $P_n(x) := \widehat{P}_n^{(\alpha-3,\beta-4)}$ and $Q_n(x) := \widehat{P}_n^{(\alpha-2,\beta)}$, for $\alpha>2$ and $\beta>3$.

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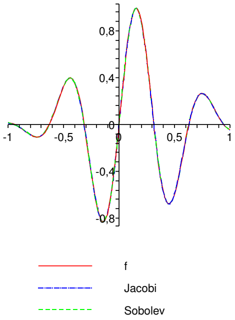
They compute the series of $f(x) := e^{-3(x-\frac{1}{10})^2}\sin(10x)$ in $[-1,1]$ with $(\alpha,\beta) = (4,5)$, $\lambda = 0.1$.

f vs Sf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f
Jacobi
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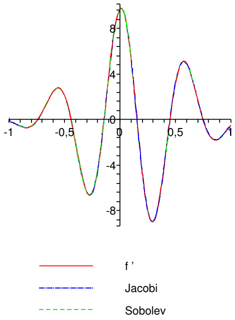
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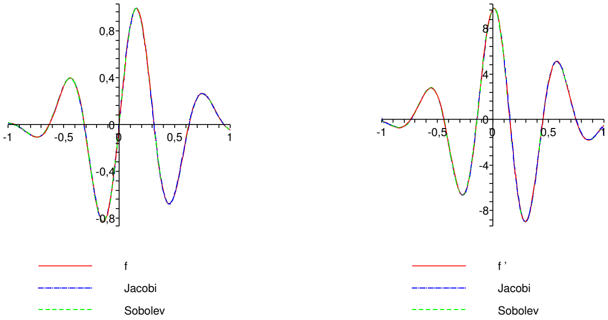
Sobolev

f vs **Sf**



0,8
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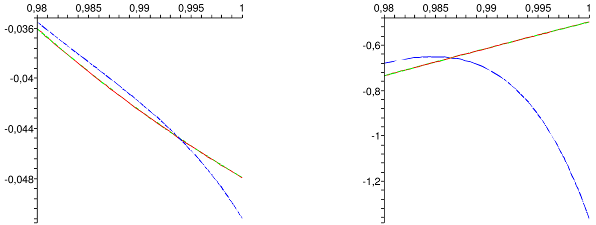
Sobolev
f '
Jacobi
Sobolev

f′ vs **(Sf)′**



0,98
0,985
0,99
0,995
1
-0,036
-0,04
-0,044
-0,048
-0,6
-0,8
-1
-1,2

Interesting application in the discrete case

THIS IS AN OPEN PROBLEM

Why?

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Why?

1. We need a nice example of discrete-coherent measures
2. We need nice examples of discrete-Sobolev OP
3. We need to build nice examples of semiclassical OP

Conclusions

1. We give a complete description of the D_ν-semiclassical discrete orthogonal polynomials in the framework of (M, N)-D_ν-coherence of order (m, k).

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2. We show an interesting application in connection with the discrete Δ-Sobolev and D_q-Sobolev orthogonal polynomials.

1. We give a complete description of the D_ν-semiclassical discrete orthogonal polynomials in the framework of (M, N)-D_ν-coherence of order (m, k).
2. We show an interesting application in connection with the discrete Δ-Sobolev and D_q-Sobolev orthogonal polynomials.
3. We have an interesting open problem ...

Our results will be appear in JCAM (avaliable online already in June 2014)

R. Alvarez-Nodarse, J. Petronilho, N. C. Pinzon-Cortes, R. Sevinik-Adıguzel, On linearly related sequences of difference derivatives of discrete orthogonal polynomials.

http://dx.doi.org/10.1016/j.cam.2014.06.018

Una polémica muy "antigua"

¿Qué Matemáticas son las que valen?

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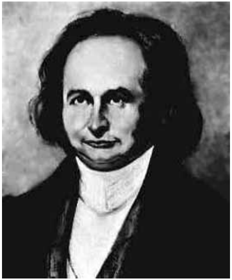
VS



Fourier: El estudio profundo de la naturaleza es el campo más fértil para los descubrimientos matemáticos ...

Jacobi: ... pero un científico como él debería saber que el único objeto de la ciencia es rendir *honor al espíritu humano* ...

VS

Lobachevsky: No hay rama de la Matemática, por abstracta que sea, que no se aplique algún día a los fenómenos del mundo real.

Hardy: Nunca he hecho nada "útil". Ningún descubrimiento mío ha supuesto [...] la más mínima alteración en el bienestar del mundo.

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Curiosidad: Resultados de Hardy se usan en criptografía y en genética.

El Teorema de Galileo

Galileo: La filosofía [natural] está escrita en ese grandioso libro que tenemos abierto ante los ojos, (quiero decir, el universo), pero no se puede entender si antes no se aprende a entender la lengua, a conocer los caracteres en los que está escrito. Está escrito en lengua matemática y sus caracteres son triángulos, círculos y otras figuras geométricas, sin las cuales es imposible entender ni una palabra; sin ellos es como girar vanamente en un oscuro laberinto.

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$\Leftarrow$ El mundo que nos rodea nos proporciona magníficos problemas matemáticos en los que trabajar.

Dehesa, Sierra, ... "Phys$\mapsto$Math" ♣ Medina, Roncal ... "Math$\mapsto$Phys"

Axioma

The tools of OP & SF are very useful for solving problems in Math. Phys.

The tools of OP & SF are very useful for solving problems in Math. Phys. which appear in **“real” problems**!

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Una buena reflexión para estos tiempos aciagos

José Echegaray
Novel de literatura de 1904

La ciencia pura es como la soberbia de oro y grana que se dilata en Occidente, entre destellos de luz y matices maravillosos: no es ilusión, es el resplandor, la hermosura de la verdad. Pero esa nube se eleva, el viento la arrastra sobre los campos y ya toma tintas más obscuras y más severas; es que va a la faena y cambia sus trajes de fiesta, digámoslo así, por la blusa de trabajo. Y entonces se condensa en lluvia, y riega las tierras, y se afana en el terruño, y prepara la futura cosecha, y al fin da a los hombres el pan nuestro de cada día. Lo que empezó por hermosura para el alma y para la inteligencia, concluye por ser alimento para la pobre vida corporal.

Academia de Ciencias, sesión solemne del 12 de marzo de 1916.

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The End

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