Orthogonal systems and semigroups

José Luis Torrea

Orthonet
Logroño, 23 Febrero 2013

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$$\underbrace{=}_{(z=re^{i\theta})} 1 + \sum_{k>0} a_k \, z^k + \sum_{k>0} r^k a_{-k} \, \bar{z}^k = U(z)$$

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Harmonic.

$$\left(r^2 \partial_r^2 + r\partial_r + \partial_\theta^2\right) P_r f(\theta) = 0 \quad \Longleftrightarrow \quad \partial_z \, \partial_{\bar{z}} \, U = 0$$

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$(r\partial_r)^* = -r\partial_r$ with respect to $d\mu(r) = \frac{dr}{r}$ en $[0,1]$.

$(\partial_\theta)^* = -\partial_\theta$ with respect to a $d\theta$.

Decomposition:

$$r^2\partial_r^2 + r\partial_r + \partial_\theta^2 = -\Big[(r\partial_r)^*(r\partial_r) + (\partial_\theta)^*(\partial_\theta)\Big]$$

Conjugate harmonic function (Fourier series)

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Cauchy–Riemann equations (Fourier series)

$$P_r f(\theta) = \sum_k r^{|k|} a_k\, e^{ik\theta} = 1 + \sum_{k>0} a_k\, z^k + \sum_{k>0} r^k a_{-k}\, \bar{z}^k = U(z)$$

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$$\partial_\theta \, (P_r f)(\theta) = -r\partial_r (Q_r f)(\theta)$$

$$r\partial_r \, (P_r f)(\theta) = \partial_\theta \, (Q_r f)(\theta) \quad \Big(\text{i.e. } (r\partial_r)^* \, (P_r f)(\theta) = (\partial_\theta)^* \, (Q_r f)(\theta)\Big)$$

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Orthogonal polynomials (Hermite)

Hermite Polynomials on $\mathbb{R}$:

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

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There is a kernel :

$$P_t f(x) = \sum_{n=0}^{\infty} e^{-tn} a_n H_n(x) = \sum_{n=0}^{\infty} e^{-tn} \left[\int_{\mathbb{R}} f(y) H_n(y) e^{-y^2} dy \right] H_n(x)$$

$$= \int_{\mathbb{R}} \left[\sum_{n=0}^{\infty} e^{-tn} H_n(y) H_n(x) \right] f(y) e^{-y^2} dy = \int_{\mathbb{R}} K_t(x,y) f(y) e^{-y^2} dy \ \ldots$$

Literature

B. Muckenhoupt, Poisson integrals for Hermite and Laguerre expansions, *Trans. Amer. Math. Soc.* **118** (1965), 17-92.

B. Muckenhoupt, Hermite conjugate expansions, *Trans. Amer. Math. Soc.* **139** (1969), 244-260.

B. Muckenhoupt and E. M. Stein Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.* **118** (1965), 17-92.

Literature

The obvious Poisson integral for a function $f(y)$ with Hermite expansion $\sum a_n H_n(y)$ is the function $g(r, y)$ with Hermite expansion $\sum r^n a_n H_n(y)$, $0 \leqq r < 1$.

application of the general theorem in §2. An alternate Poisson integral, $f(x, y)$, is also mentioned. If $f(y)$ has the Hermite expansion given above, $f(x, y)$ is the function which for fixed $x > 0$ has the expansion $\sum a_n \exp[-(2n)^{1/2}x]H_n(y)$. The theorems proved for g are immediately applicable to this since there is a simple relation between it and g. Like the ordinary Poisson integral, $f(x, y)$ satisfies a second order elliptic differential equation. In fact, $f_{11}(x, y) + f_{22}(x, y) - 2yf_2(x, y) = 0$. This makes $f(x, y)$ a more reasonable Poisson integral and makes it possible to define conjugate functions for Hermite expansions. These conjugate functions will be treated in another paper.

It was shown in [2] that

$$\frac{\partial^2 f(x, y)}{\partial x^2} + \exp(y^2)\frac{\partial}{\partial y}\left(\exp(-y^2)\frac{\partial f(x, y)}{\partial y}\right) = 0. \tag{1.1}$$

Similarly, it will be shown here that

$$\frac{\partial^2 \tilde{f}(x, y)}{\partial x^2} + \frac{\partial}{\partial y}\left[\exp(y^2)\frac{\partial}{\partial y}(\exp(-y^2)\tilde{f}(x, y))\right] = 0 \tag{1.2}$$

and that the analogues of the Cauchy-Riemann equations

$$\frac{\partial f(x, y)}{\partial x} = \exp(y^2)\frac{\partial}{\partial y}(\exp(-y^2)\tilde{f}(x, y)) \tag{1.3}$$

Understanding Muckenhoupt

Two alternatives following Muckenhoupt

$$\text{“}\, P_t f(x) = \sum_n e^{-t2n} a_n H_n(x) \,\text{”} \qquad \text{versus} \qquad P_t f(x) = \sum_n e^{-t\sqrt{2n}} a_n H_n(x).$$

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The second satisfies

$$\begin{aligned}\left(\partial_t^2 + \partial_x^2 - 2x\partial_x\right)(P_t f)(x) &= \left(\partial_t^2 + (\partial_x - 2x)\partial_x\right)(P_t f)(x)\\ &= -\Big[(\partial_t)^*(\partial_t) + (\partial_x)^*\partial_x\Big](P_t f)(x) = 0\end{aligned}$$

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$$(\partial_x)^* = -(\partial_x - 2x)$$

Adjoint with respect to measure $d\gamma(x) = e^{-x^2}dx$

Moreover

$$Q_t f(x) = \sum_n \sqrt{2n}\, e^{-t\sqrt{2n}}\, a_n\, H_{n-1}(x)$$

satisfies

$$\left(\partial_t^2 + \partial_x(\partial_x - 2x)\right)(Q_t f)(x) = -\left[(\partial_t)(\partial_t)^* + \partial_x(\partial_x)^*\right](Q_t f)(x) = 0$$

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Cauchy–Riemann equations (Hermite)

$$\partial_x P_t f(x) = -\partial_t Q_t f(x)$$

$$\partial_t P_t f(x) = (\partial_x - 2x) Q_t f(x), \quad \Big(\text{i.e. } (\partial_t)^* P_t f(x) = (\partial_x)^* Q_t f(x)\Big)$$

Common path?

In all the cases,

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$\lim_{t\to 0} Q_t f(x)$ drives to an important operator:

(1) $\lim_{t\to 0} -i \sum_{k\neq 0} \operatorname{sign} k\, r^{|k|} a_k\, e^{ik\theta} = -i \sum_{k\neq 0} \operatorname{sign} k\, a_k\, e^{ik\theta}$ (Conjugate function).

(2) $\lim_{t\to 0} -i \int_{\mathbb{R}} \operatorname{sign} \xi e^{-t|\xi|} \widehat{f}(\xi) e^{i\xi x} d\xi = \int_{\mathbb{R}} \widehat{Hf}(\xi) e^{i\xi x} d\xi$ (Hilbert transform).

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$$¿ \quad \partial_x \int_0^\infty P_s f(x)\, ds \quad ?$$

Diffusion semigroups

E. Stein, *Topics in Harmonic Analysis Related to the Littlewood-Paley theory*, Princeton, 1970.

$(\mathcal{M}, d\mu)$ measure space . $\{T_t\}_{t>0} : L^2 \to L^2$:

- $T_{t_1+t_2} = T_{t_1} T_{t_2}$. $T_0 = Id$. $\lim_{t\to 0} T_t f = f$ in L^2.
- $\|T_t f\|_p \le \|f\|_p$, $(1 \le p \le \infty)$. Contraction.
- T_t selfadjoint in L^2.
- $T_t f \ge 0$ si $f \ge 0$. Positivity.
- $T_t 1 = 1$. Markov.

First example of semigroup. Classical heat equation

The example of diffusion semigroup in $L^2(\mathbb{R})$:

$$T_t f(x) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy$$

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For good functions f we have:

$$\partial_t(T_t f(x)) = \partial_x^2(T_t f(x)) = \Delta_x(T_t f(x))$$

In symbols

$$T_t f(x) = e^{t\Delta} f(x) \quad \text{“heat semigroup”}$$

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Interesting remark

$$\widehat{T_t f}(\xi) = e^{-t|\xi|^2} \widehat{f}(\xi)$$

Second example of diffusion semigroup. Orthogonal polynomials

Ilustration. L with eigenfunctions $\{\phi_k\}_k$ and eigenvalues $\{\lambda_k\}_k$

$$e^{-tL}\phi_k(x) = e^{-t\lambda_k}\phi_k(x), \qquad e^{-tL}\Big(\sum_k c_k\phi_k\Big)(x) = \sum_k e^{-t\lambda_k}c_k\phi_k(x)$$

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Hermite polynomials case. $L = \partial_x^2 + 2x\partial_x, \quad LH_n = 2nH_n$. Then

$$e^{-tL}\Big(\sum_n a_nH_n\Big)(x) = \sum_n e^{-t2n}a_nH_n(x)$$

(remember Muckenhoupt)

Third example of Poisson semigroup

Formula for Gamma function:

$$e^{-t\sqrt{\lambda}} = \frac{t}{2\sqrt{\pi}} \int_0^{\infty} \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}} e^{-s\lambda} ds, \quad \lambda > 0.$$

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$e^{-t\sqrt{L}} f$: Poisson semigroup

If L can be decompose $L = (\partial_x)^*(\partial_x)$, we get the **conjugate harmonic function**:

$$Q_t f = -\int_t^\infty \partial_s Q_s f \, ds = \int_t^\infty \partial_x P_s f \, ds$$

Determination of kernels

If we know e^{-tL} then we know $e^{-t\sqrt{L}}$.

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$$\frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{3/2}}\, e^{-sL}(x,y)\,ds = \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}}\frac{1}{\sqrt{4\pi s}}e^{-\frac{|x-y|^2}{4s}}\,ds$$

$$\underbrace{=}_{\left(\frac{t^2+|x-y|^2}{4s}=u\right)} \frac{t}{\pi}\int_0^\infty \frac{1}{(t^2+|x-y|^2)}\,e^{-u}\,du$$

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Conclusion. If we know the kernel of the heat semigroup, we know the kernel of the Poisson semigroup.

“Riesz Transforms”

Gamma function firmula: $\lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha e^{-\lambda t} \frac{dt}{t}, \quad \lambda, \alpha > 0.$

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Hence,

$$L^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha \, e^{-tL} f \, \frac{dt}{t}$$

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Hence,

$$L^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^\alpha \, e^{-tL} f \, \frac{dt}{t}$$

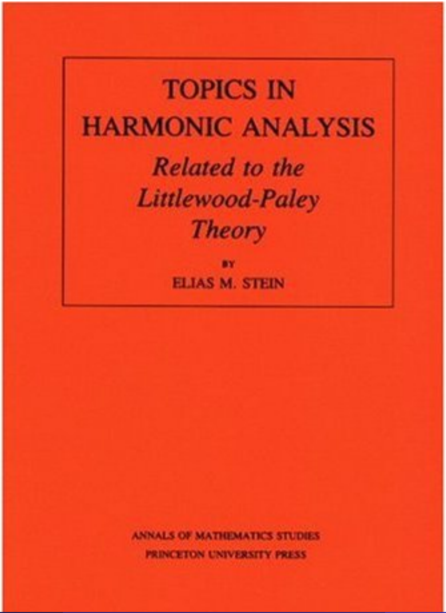
$$L^{-1/2} f = (\sqrt{L})^{-1} f = \frac{1}{\Gamma(1)} \int_0^\infty t^1 \, e^{-t\sqrt{L}} f \, \frac{dt}{t} = \int_0^\infty e^{-t\sqrt{L}} f \, dt$$

Going back to our slide (¿ ?)

$$\lim_{t\to 0} Q_t f = \lim_{t\to 0} - \int_t^\infty \partial_s Q_s f(x) ds = \partial_x \int_0^\infty P_s f(x) ds = \partial_x L^{-1/2} f$$

Stein knew everything!.

Stein knew everything!. Look into the red book. **¡1970!**



Stein knew it . . .

We assume that G is a non-compact, connected, Lie group. We let $X_1, X_2, \ldots, X_n$ be a basis for the (left-invariant) Lie algebra, considered as first-order differential operators on G. We set

$$\Delta^+ = \Sigma\, a_{ij} X_i X_j$$

where $\{a_{ij}\}$ is *any* real symmetric positive definite matrix. (More specific choices of the $\{a_{ij}\}$ will be made later.) Our first object is to consider the heat-diffusion semigroup $T_+^t = e^{t\Delta^+}$.

The definition of the Riesz transforms can be given symbolically as

$$R_i(f) = \bar{X}_i(-\Delta)^{-1/2} f\,.$$

Boundedness in L^2 of Riesz transforms

$$L = \partial_x^* \partial_x$$

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Eigenvalue case: $\psi_k = \partial_x (L)^{-1/2} \phi_k$

$$\int \psi_k \psi_\ell d\mu = \int \left(\partial_x (L)^{-1/2} \phi_k\right)\left(\partial_x (L)^{-1/2} \phi_\ell\right) d\mu$$

$$= \int \left(\partial_x^* \partial_x (L)^{-1/2} \phi_k\right)\left((L)^{-1/2} \phi_\ell\right) d\mu = \int \left(L (L)^{-1/2} \phi_k\right)\left((L)^{-1/2} \phi_\ell\right) d\mu$$

$$= \int \left((L)^{1/2} \phi_k\right)\left((L)^{-1/2} \phi_\ell\right) d\mu = \lambda_k^{1/2} \lambda_\ell^{-1/2} \int \phi_k \phi_\ell \, d\mu$$

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Spectral theorem:

$$\begin{aligned} \langle \partial_x (L)^{-1/2} f, \partial_x (L)^{-1/2} f \rangle &= \langle \partial_x^* \partial_x (L)^{-1/2} f, (L)^{-1/2} f \rangle \\ &= \langle (L)^{1/2} f, (L)^{-1/2} f \rangle = \langle f, f \rangle \end{aligned}$$

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General procedure in Harmonic Analysis. Once we know the L^2-boundedness, the kernel is used to get L^p boundedness.

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$$\begin{aligned}\partial_x L^{-1/2} f(x) &= \partial_x \int_0^\infty e^{-tL} f(x)\, t^{1/2}\, \frac{dt}{t} = \int_0^\infty \partial_x e^{-tL} f(x)\, t^{1/2}\, \frac{dt}{t} \\ &= \int_0^\infty \partial_x \int_{\mathbb{R}^n} e^{-tL}(x,y) f(y)\, dy\, t^{1/2}\, \frac{dt}{t} \\ &= \int_{\mathbb{R}^n} \left(\int_0^\infty \partial_x\, e^{-tL}(x,y)\, t^{1/2}\, \frac{dt}{t} \right) f(y)\, dy \\ &= \int_{\mathbb{R}^n} K(x,y) f(y)\, dy.\end{aligned}$$

Possible Applications – “a priori” estimates

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$$L = \sum_i (\partial_i)^* \partial_i$$

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$$\partial_i \partial_j u = \partial_i \partial_j L^{-1} f \in L^p$$

Application – Sobolev Spaces

Assume

$$\|f\|_{L^p} \sim \|\partial L^{-1/2} f\|_{L^p}.$$

Equivalently

$$\|L^{1/2} g\|_{L^p} \sim \|\partial g\|_{L^p}.$$

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Sobolev spaces

$$W_L^{k,p} = \{f \in L^p \ : \ \partial^k f \in L^p\} = \{f \in L^p \ : \ L^{k/2} f \in L^p\}$$

¡Muchas gracias!