Uncertainty relations

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Primer Encuentro de la Red de Polinomios Ortogonales
y Teoría de Aproximación
ORTHONET 2013, 22-23 February 2013

Introduction: Uncertainty Relations

The position-momentum uncertainty principle is one of the most prominent differences between classical and quantum physics.
Its most celebrated representation is the Heisenberg uncertainty relation:

$$V[\rho]V[\gamma] \geq \frac{D^2}{4},$$

where D is the dimensionality of the space, and

$$V[\rho] = \langle \mathbf{x}^2 \rangle - \langle \mathbf{x} \rangle^2$$

is the variance of the probability density function $\rho(\mathbf{x})$ describing the quantum system in the position space. And analogously $V[\gamma]$ for $\gamma(\mathbf{p})$ in the momentum space.
Notation: $\langle f(\mathbf{x}) \rangle = \int_{\mathbb{R}^D} f(\mathbf{x})\rho(\mathbf{x})d\mathbf{x}$.

Momentum space: Fourier transform

Wavefunction and density in the position space:

$$\psi(\mathbf{x}) \quad \rightarrow \quad \rho(\mathbf{x}) = |\psi(\mathbf{x})|^2.$$

Wavefunction and density in the momentum space:

$$\tilde{\psi}(\mathbf{p}) = \int e^{-i\mathbf{x}\cdot\mathbf{p}}\psi(\mathbf{x})d\mathbf{x} \quad \rightarrow \quad \gamma(\mathbf{p}) = |\tilde{\psi}(\mathbf{p})|^2.$$

Entropic uncertainty relations

Apart from the Heisenberg uncertainty relation based on the variances of the densities, there are uncertainty relations expressed in terms of other information measures:

- Shannon entropy [Bialynicki-Birula et al., Commun. Math. Phys. (1975)]

$$S[\rho] = -\int_{\mathbb{R}^D} \rho(\mathbf{x}) \ln \rho(\mathbf{x}) d\mathbf{x}, \quad S[\gamma] = -\int_{\mathbb{R}^D} \gamma(\mathbf{p}) \ln \gamma(\mathbf{p}) d\mathbf{p}$$

$$S[\rho] + S[\gamma] \geq D(1 + \ln \pi)$$

- Rényi entropy [Bialynicki-Birula, Phys. Rev. A (2006)]

$$R_\alpha[\rho] = \frac{1}{1-\alpha} \ln \int_{\mathbb{R}^D} [\rho(\mathbf{x})]^\alpha d\mathbf{x}, \quad R_\beta[\gamma] = \frac{1}{1-\beta} \ln \int_{\mathbb{R}^D} [\gamma(\mathbf{p})]^\beta d\mathbf{p}$$

$$R_\alpha[\rho] + R_\beta[\gamma] \geq \frac{1}{2(\alpha-1)} \ln \frac{\alpha}{\pi} + \frac{1}{2(\beta-1)} \ln \frac{\beta}{\pi}, \qquad \frac{1}{\alpha} + \frac{1}{\beta} = 2$$

Entropic uncertainty relations

- Tsallis entropy [Rajagopal, Phys. Lett. A (1995)]

$$T_\alpha[\rho] = \frac{1}{\alpha - 1}\left(1 - \int_{\mathbb{R}^D} [\rho(\mathbf{x})]^\alpha d\mathbf{x}\right), \quad T_\beta[\gamma] = \frac{1}{\beta - 1}\left(1 - \int_{\mathbb{R}^D} [\gamma(\mathbf{p})]^\beta d\mathbf{p}\right)$$

$$\frac{[1 + (1-\alpha)T_\alpha[\rho]]^{\frac{1}{2\alpha}}}{[1 + (1-\beta)T_\beta[\gamma]]^{\frac{1}{2\beta}}} \geq \left(\frac{\alpha}{\pi}\right)^{-\frac{D}{4\alpha}} \left(\frac{\beta}{\pi}\right)^{\frac{D}{4\beta}}, \qquad \frac{1}{\alpha} + \frac{1}{\beta} = 2$$

- Fisher information [Sánchez-Moreno et al., J. Phys. A (2011)]

$$F[\rho] = \int_{\mathbb{R}^D} \frac{|\nabla \rho(\mathbf{x})|^2}{\rho(\mathbf{x})} d\mathbf{x}, \quad F[\gamma] = \int_{\mathbb{R}^D} \frac{|\nabla \gamma(\mathbf{p})|^2}{\gamma(\mathbf{p})} d\mathbf{p}$$

$$F[\rho]F[\gamma] \geq 4D^2$$

Improvements to the uncertainty relations

- The Schrödinger-Robertson uncertainty relation [Robertson, Phys. Rev. (1930)]:

$$V[\rho]V[\gamma] - \frac{1}{4}\left(\langle \mathbf{xp} + \mathbf{px}\rangle - 2\langle \mathbf{x}\rangle\langle \mathbf{p}\rangle\right)^2 \geq \frac{1}{4}.$$

- Taking into account the purity μ of the quantum system [Dodonov, J. Opt. B (2012)]:

$$V[\rho]V[\gamma] - \frac{1}{4}\left(\langle \mathbf{xp} + \mathbf{px}\rangle - 2\langle \mathbf{x}\rangle\langle \mathbf{p}\rangle\right)^2 \geq \frac{1}{4}\Phi(\mu).$$

- Taking into account the gaussianity g of the densities [Mandilara and Cerf, Phys. Rev. A (2012)]:

$$V[\rho]V[\gamma] - \frac{1}{4}\left(\langle \mathbf{xp} + \mathbf{px}\rangle - 2\langle \mathbf{x}\rangle\langle \mathbf{p}\rangle\right)^2 \geq \frac{1}{4}f(g).$$

- Taking into account the spherical symmetry of the system (central systems).

Central systems

A quantum central system satisfies a Schrödinger equation

$$-\frac{1}{2}\nabla^2\psi(\mathbf{x}) + V(\mathbf{x})\psi(\mathbf{x}) = E\psi(\mathbf{x}),$$

where the potential is spherically symmetric:

$$V(\mathbf{x}) = V(r), \qquad r = \|\mathbf{x}\|.$$

Then, the wavefunction $\psi(\mathbf{x})$ can be factorized in hyperspherical coordinates:

$$\psi_{E,l,\{\mu\}}(\mathbf{x}) = R_{E,l}(r)\mathcal{Y}_{l,\{\mu\}}(\Omega_{D-1}).$$

The angular part is a hyperspherical harmonic independent of $V(r)$ that depends on a set of quantum angular numbers

$$(l,\{\mu\}) = (l \equiv \mu_1, \mu_2, \ldots, \mu_{D-1} \equiv m), \quad \mu_i \in \mathbb{Z},$$

$$l \equiv \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{D-2} \geq |\mu_{D-1}| \equiv |m| \geq 0.$$

Central systems: Hankel transform

The density is factorized

$$\rho_{E,l,\{\mu\}}(\mathbf{x}) = \left|\psi_{E,l,\{\mu\}}(\mathbf{x})\right|^2 = |R_{E,l}(r)|^2 |\mathcal{Y}_{l,\{\mu\}}(\Omega_{D-1})|^2$$

In terms of reduced radial functions:

$$\rho_{E,l,\{\mu\}}(\mathbf{x}) = \frac{|u_{E,l}(r)|^2}{r^{D-1}} |\mathcal{Y}_{l,\{\mu\}}(\Omega_{D-1})|^2$$

In the momentum space:

$$\gamma_{E,l,\{\mu\}}(\mathbf{p}) = \frac{|\tilde{u}_{E,l}(p)|^2}{p^{D-1}} |\mathcal{Y}_{l,\{\mu\}}(\Omega_{D-1})|^2,$$

where $\tilde{u}_{E,l}(p)$ is the Hankel transform of $u_{E,l}(r)$:

$$\tilde{u}_{E,l}(p) = (-i)^l \int_0^\infty \sqrt{rp} J_{l+\frac{D}{2}}(rp) u_{E,l}(r) dr.$$

Uncertainty relations in central systems

Heisenberg-like uncertainty relation:

- General relation:

$$\langle \mathbf{x}^2 \rangle \langle \mathbf{p}^2 \rangle \geq \frac{D^2}{4}$$

- For central systems [Sánchez-Moreno et al, New J. Phys. (2006)]:

$$\langle \mathbf{x}^2 \rangle \langle \mathbf{p}^2 \rangle \geq \left(l + \frac{D}{2} \right)^2$$

Fisher-information-based uncertainty relation:

- General relation:

$$F[\rho] F[\gamma] \geq 4D^2$$

- For central systems [Dehesa et al, J. Phys. A (2007)]:

$$F[\rho] F[\gamma] \geq 16 \left(1 - \frac{2|m|}{2l + D - 2} \right)^2 \left(l + \frac{D}{2} \right)^2$$

General Shannon-entropy-based uncertainty relation

The Shannon entropies of position and momentum continuous probability distributions $\rho(\mathbf{x})$ and $\gamma(\mathbf{p})$ fulfill the entropic uncertainty relation [Bialynicki-Birula (1975)]

$$S[\rho] + S[\gamma] \geq D(1 + \ln \pi)$$

Our aim is to find an uncertainty relation for central systems.

[Bialynicki-Birula (1975)] I. Bialynicki-Birual and J. Mycielski, *Uncertainty relations for information entropy in wave mechanics*, Commum. Math. Phys. **44** (1975) 129-132.

Position Shannon entropy in central systems

In [Rudnicki et al. (2012)] the density is expressed as

$$\rho(\mathbf{x}) = \frac{|u(r)|^2}{r^{D-1}} |\mathcal{Y}_{l,\{\mu\}}(\Omega_{D-1})|^2,$$

so that $\int_0^\infty |u(r)|^2 dr = 1$.
If $\omega(r) = |u(r)|^2$, then

$$S[\rho] = S[\omega] + (D-1)\langle \ln r \rangle + S(\mathcal{Y}_{l,\{\mu\}}).$$

[Rudnicki et al. (2012)] Ł. Rudnicki, P. Sánchez-Moreno, J.S. Dehesa, *The Shannon-entropy-based uncertainty relation for D-dimensional central potentials*, J. Phys. A **45** (2012) 225303.

Position Shannon entropy in central systems

In the position space:

$$S[\rho] = S[\omega] + (D-1)\langle \ln r \rangle + S(\mathcal{Y}_{l,\{\mu\}}),$$

where

$$S[\omega] = -\int_0^\infty \omega(r) \ln \omega(r) dr,$$

$$\langle \ln r \rangle = \int_0^\infty \omega(r) \ln r dr,$$

$$S(\mathcal{Y}_{l,\{\mu\}}) = -\int_{S^{D-1}} \left|\mathcal{Y}_{l,\{\mu\}}(\Omega_{D-1})\right|^2 \ln \left|\mathcal{Y}_{l,\{\mu\}}(\Omega_{D-1})\right|^2 d\Omega_{D-1}.$$

Momentum Shannon entropy in central systems

Analogously, in the momentum space:

$$S[\gamma] = S[\tilde{\omega}] + (D-1)\langle \ln p \rangle + S(\mathcal{Y}_{l,\{\mu\}}),$$

where

$$S[\tilde{\omega}] = -\int_0^\infty \tilde{\omega}(p) \ln \tilde{\omega}(p) dp, \quad \tilde{\omega}(p) = |\tilde{u}(p)|^2,$$

$$\langle \ln p \rangle = \int_0^\infty \tilde{\omega}(p) \ln p dp,$$

$$S(\mathcal{Y}_{l,\{\mu\}}) = -\int_{S^{D-1}} \left|\mathcal{Y}_{l,\{\mu\}}(\Omega_{D-1})\right|^2 \ln \left|\mathcal{Y}_{l,\{\mu\}}(\Omega_{D-1})\right|^2 d\Omega_{D-1}.$$

Entropy sum for central systems

$$S[\rho] + S[\gamma] = S[\omega] + S[\tilde{\omega}] + (D-1)(\langle \ln r \rangle + \langle \ln p \rangle) + 2S(\mathcal{Y}_{l,\{\mu\}})$$

Entropy sum for central systems

Dependent on $V(r)$ — Independent of $V(r)$

$$S[\rho] + S[\gamma] = \underbrace{S[\omega] + S[\tilde{\omega}]}_{\text{Dependent on } V(r)} + (D-1)\big(\underbrace{\langle \ln r \rangle + \langle \ln p \rangle}_{\text{Dependent on } V(r)}\big) + \underbrace{2S(\mathcal{Y}_{l,\{\mu\}})}_{\text{Independent of } V(r)}$$

Entropy sum for central systems

Dependent on $V(r)$ Independent of $V(r)$

$$S[\rho] + S[\gamma] = S[\omega] + S[\tilde{\omega}] + (D-1)(\langle \ln r \rangle + \langle \ln p \rangle) + 2S(\mathcal{Y}_{l,\{\mu\}})$$

To be bounded To be bounded

Entropic U.R. for the reduced radial wavefunctions

$\tilde{u}(p)$ is the Hankel transform of $u(r)$. Then, [De Carli (2008)]

$$\left(\int_0^\infty |\tilde{u}(p)|^q dp\right)^{\frac{1}{q}} \leq C(q,q',\nu)\left(\int_0^\infty |u(r)|^{q'} dr\right)^{\frac{1}{q'}}$$

with $1 < q' \leq 2$, and $\frac{1}{q'} + \frac{1}{q} = 1$, and

$$C(q,q',\nu) = \frac{A(q',\nu)}{A(q,\nu)}, \quad A(q,\nu) = 2^{\frac{1}{2q}} \frac{q^{\frac{1}{2}\left(\nu+\frac{1}{2}+\frac{1}{q}\right)}}{\Gamma\left(\frac{q}{2}\left(\nu+\frac{1}{2}\right)+\frac{1}{2}\right)^{\frac{1}{q}}}.$$

[De Carli (2008)] L. De Carli, *On the $L^p - L^q$ norm of the Hankel transform and related operators*, J. Math. Anal. Appl. **348** (2008) 366-382.

Entropic U.R. for the reduced radial probability densities

Rewriting the previous inequality for the reduced radial probability densities $\omega(r)$ and $\tilde{\omega}(p)$, with $\nu = l + \frac{D}{2} - 1$:

$$\left(\int_0^\infty (\tilde{\omega}(p))^\alpha dp\right)^{\frac{1}{2\alpha}} \le C(2\alpha, 2\beta, \nu)\left(\int_0^\infty (\omega(r))^\beta dr\right)^{\frac{1}{2\beta}}$$

that yields

$$\underbrace{\frac{1}{1-\alpha}\ln\left(\int_0^\infty (\tilde{\omega}(p))^\alpha dp\right)}_{R_\alpha[\tilde{\omega}]} \ge \frac{2\alpha \ln C(2\alpha, 2\beta, \nu)}{1-\alpha} - \underbrace{\frac{1}{1-\beta}\ln\left(\int_0^\infty (\omega(r))^\beta dr\right)}_{R_\beta[\omega]}$$

$$R_\beta[\omega] + R_\alpha[\tilde{\omega}] \ge \frac{2\alpha \ln[A(2\alpha, \nu)]}{\alpha - 1} + \frac{2\beta \ln[A(2\beta, \nu)]}{\beta - 1}$$

Shannon U.R. for the reduced radial probability densities

Taking the limit $\alpha \to 1$, with $\beta = \frac{\alpha}{(2\alpha-1)}$, the previous inequality for Rényi entropies yields

$$S[\omega] + S[\tilde{\omega}] \geq 2l + D + 2\ln\left(\frac{\Gamma\left(l+\frac{D}{2}\right)}{2}\right) - (2l+D-1)\psi\left(l+\frac{D}{2}\right)$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ denotes the digamma function.

Logarithmic uncertainty relation for general systems

There exist a previous logarithmic uncertainty relation [Beckner (1995)]:

$$\langle \ln r \rangle + \langle \ln p \rangle \geq \psi\left(\frac{D}{4}\right) + \ln 2.$$

But it is not sharp enough for the Shannon-entropy-based uncertainty relation.

[Beckner (1995)] W. Beckner, *Pitt's inequality and the uncertainty principle*, Proc. Am. Math. Soc. **123** (1995) 1897-1905.

Logarithmic uncertainty relation for central systems

To prove an improved logarithmic relation for central potentials, we begin with this relation [Omri (2011)]:

$$\int_0^\infty |f(r)|^2 \ln r \, d\lambda_\mu(r) + \int_0^\infty |\tilde{f}(p)|^2 \ln p \, d\lambda_\mu(p) \geq \left(\psi\left(\frac{\mu+1}{2}\right) + \ln 2 \right) N_\mu,$$

where

$$d\lambda_\mu(r) = \frac{r^{2\mu+1}}{2^\mu \Gamma(\mu+1)} dr, \quad N_\mu = \int_0^\infty |f(r)|^2 d\lambda_\mu(r),$$

and $\tilde{f}$ is the Hankel transform of f.

[Omri (2011)] S. Omri, *Logarithmic uncertainty principle for the Hankel transform*, Integ. Trans. Spec. Funct. **22** (2011) 655-670.

Logarithmic uncertainty relation for central systems

If we take $f(r) = r^{-l-\frac{D-1}{2}} u(r)$ and $\mu = l + \frac{D}{2} - 1$, we obtain that

$$\langle \ln r \rangle + \langle \ln p \rangle \geq \psi \left(\frac{2l + D}{4} \right) + \ln 2$$

that reduces to the Beckner inequality for $l = 0$:

$$\langle \ln r \rangle + \langle \ln p \rangle \geq \psi \left(\frac{D}{4} \right) + \ln 2.$$

Shannon-entropy-based uncertainty relation for central potentials

Taking into account the previous results, we obtain the following Shannon-entropy-based uncertainty relation for D-dimensional central potentials:

$$S[\rho]+S[\gamma]=S[\omega]+S[\tilde{\omega}]+(D-1)(\langle\ln r\rangle+\langle\ln p\rangle)+2S(\mathcal{Y}_{l,\{\mu\}})\geq B_{l,\{\mu\}}$$

$$B_{l,\{\mu\}}=2l+D+2\ln\left(\frac{\Gamma\left(l+\frac{D}{2}\right)}{2}\right)-(2l+D-1)\psi\left(l+\frac{D}{2}\right)+(D-1)\left(\psi\left(\frac{2l+D}{4}\right)+\ln 2\right)+2S(\mathcal{Y}_{l,\{\mu\}})$$

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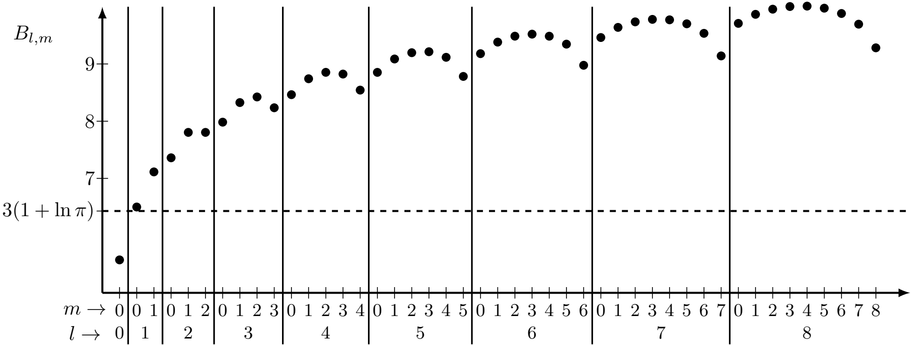
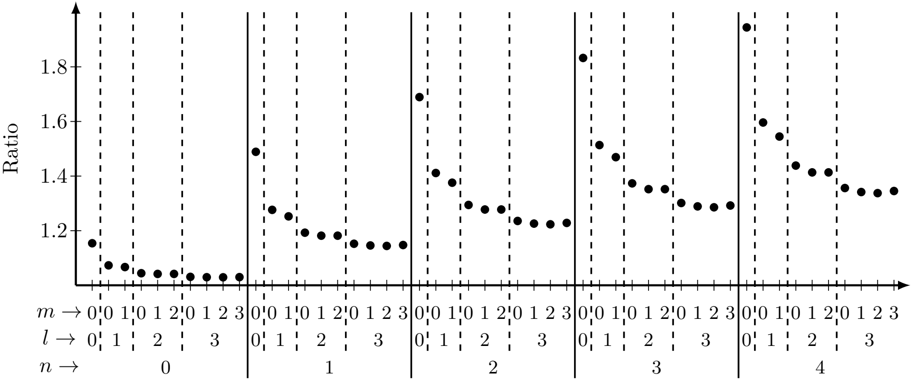
$$B_{l,\{\mu\}} = 2l + D + 2\ln\left(\frac{\Gamma\left(l+\frac{D}{2}\right)}{2}\right) - (2l+D-1)\psi\left(l+\frac{D}{2}\right) + (D-1)\left(\psi\left(\frac{2l+D}{4}\right) + \ln 2\right) + 2S(\mathcal{Y}_{l,\{\mu\}})$$

Comparison with the general bound



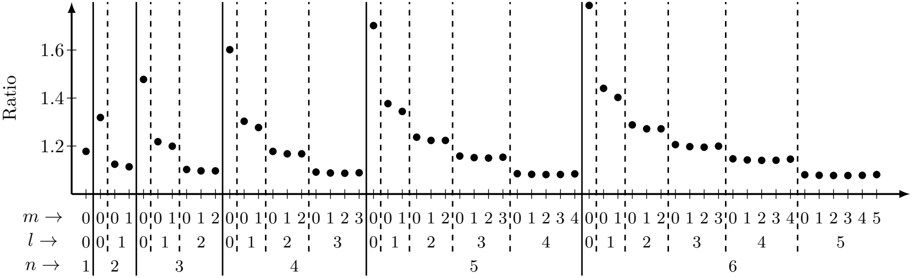
Comparison of the improved bound for central systems $B_{l,m}$ and the general Bialynicki-Birula bound for $D = 3$.

An example: The harmonic oscillador $V(r) = \frac{1}{2}r^2$



Ratio of the exact value of the sum of Shannon entropies and the new bound for the harmonic oscillator with $D = 3$.

An example: The hydrogen atom $V(r) = r^{-1}$



Ratio of the exact value of the sum of Shannon entropies and the new bound for the hydrogen atom with $D = 3$.

Summary and conclusions

- The position-momentum uncertainty relations set up inequalities involving information measures applied to the densities in the position and momentum spaces. The amplitude functions giving these densities are connected by a Fourier transform.
- These uncertainty relations are based on several information measures.
- These uncertainty relations can be improved by imposing additional conditions on the probability densities.
- When central systems are considered, the densities in the position and momentum spaces are factorized, where the radial parts are connected by a Hankel transform, and the angular parts are hyperspherical harmonics.

Summary and conclusions

- The Heisenberg-like relation for central systems improves as

$$\langle \mathbf{x}^2 \rangle \langle \mathbf{p}^2 \rangle \geq \left(l + \frac{D}{2} \right)^2$$

- The Fisher-information-based uncertainty relation for central systems is

$$F[\rho]F[\gamma] \geq 16 \left(1 - \frac{2|m|}{2l + D - 2} \right)^2 \left(l + \frac{D}{2} \right)^2$$

Summary and conclusions

- In the case of the Shannon entropy-based uncertainty relation for central systems, we have used results by De Carli and Omri to obtain an improved lower bound to the sum of Shannon entropies.

$$S[\rho]+S[\gamma]\geq 2l+D+2\ln\left(\frac{\Gamma\left(l+\frac{D}{2}\right)}{2}\right)-(2l+D-1)\psi\left(l+\frac{D}{2}\right)$$
$$+(D-1)\left(\psi\left(\frac{2l+D}{4}\right)+\ln 2\right)+2S(\mathcal{Y}_{l,\{\mu\}}).$$

Open problems: Uncertainty relations

- To find a Shannon entropy-based uncertainty relation for central systems that improves the general one by Bialynicki-Birula in all the cases.

- To improve the uncertainty relations based on the Rényi and Tsallis entropies for central systems.

Open problems: Orthogonal polynomials

- To find uncertainty relations saturated by the Rakhmanov densities associated to families of orthogonal polynomials.

Analogously to the work by Mandilara and Cerf [Phys. Rev. A (2012)] where an uncertainty relation is found that is saturated by all the eigenfunctions of the harmonic oscillator, that is by all the Rakhmanov densities of the Hermite polynomials:

$$V[\rho]V[\gamma] \geq f(g).$$

Thank you!