On positive rational interpolatory quadrature formulas on the unit circle with prescribed nodes

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Primer Encuentro de la Red de Polinomios Ortogonales y Teoría de Aproximación, ORTHONET 2013, Logroño, 22 y 23 de febrero de 2013

" To Pablo González Vera, our beloved master and friend. In memóriam."

Orthogonal Rational Functions. Aim of this talk.

- ORF: generalization of orthogonal ordinary polynomials (poles at ∞) and orthogonal Laurent polynomials (poles at $\{0, \infty\}$).
- AIM OF THIS TALK:
  1. To present a research topic between people from
     - La Laguna University, Tenerife (P. González-Vera, C. Díaz-Mendoza, R. Orive, F. Perdomo Pío, R. Cruz-Barroso).
     - K.U. Leuven, Belgium (A. Bultheel).
     - U. Lille (B. Beckermann, K. Deckers).
  2. More exactly: positive rational interpolatory quadrature formulas on the unit circle with prescribed nodes.
  3. Survey and some recent results:
     - Connection with some recent results by B. Beckermann and K. Deckers.
     - Extension to ORF of some of the results presented in: *Orthogonal Polynomials and Special Functions - a Complex Analytic Perspective. Copenhagen, Denmark, June 11-15, 2012* (P. González-Vera, C. Díaz-Mendoza and F. Perdomo Pío, R. Cruz-Barroso).

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Some works

- A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad.- **Orthogonal Rational Functions**, volume 5 of *Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 1999. (1st Bible)*
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Rational quadrature formulas on the unit circle

- $\mathring{\mu}$: positive Borel measure on $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.
- Aim: estimations of $I_{\mathring{\mu}}(f) = \int_{-\pi}^{\pi} f(z) d\mathring{\mu}(\theta)$, with $z = e^{i\theta}$, f continuous on $\mathbb{T}$.
- Quadrature formulas:

$$I_n(f) = \sum_{k=1}^{n} \lambda_k f(z_k) \quad z_j \neq z_k \;\; if \;\; j \neq k.$$

Election of the nodes and weights so that the rule is exact in a certain space of functions with dimension as large as possible.

$\rightsquigarrow$ Positive rule (positive weights): interest for convergence and stability.

Rational quadrature formulas on the unit circle

- Quadrature formulas for $I_{\mathring{\mu}}(f)$: n-point Szegő rules, positive and exact in $span\{z^k : -(n-1) \leq k \leq n-1\}$.
  $\rightsquigarrow$ One free parameter of modulus one.
  $\rightsquigarrow$ Nodal polynomial: para-orthogonal.

- Rational quadrature formulas for $I_{\mathring{\mu}}(f)$: rational n-point Szegő rules, positive and exact in spaces of rational functions.

  $\rightsquigarrow$ Rational quadratures: in many cases, f is much better approximated by rational functions, instead of L-polynomials.

  e.g. f meromorphic, with poles outside but maybe close to $\mathbb{T}$.

Rational generalization of well known concepts

- $\mathring{\mu} \rightsquigarrow$ inner product $\langle f, g\rangle_{\mathring{\mu}} = \int_{-\pi}^{\pi} f(z)\overline{g(z)}d\mathring{\mu}(\theta)$, $z = e^{i\theta}$.
- Initial data: $\mathring{\mu}$ and $\alpha = \{\alpha_k\}_{k=1}^{\infty} \subset \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.
- Blaschke factor: $\zeta_i(z) = \frac{z-\alpha_i}{1-\overline{\alpha_i}z}$,
- Blaschke products: $B_0 \equiv 1$, $B_n = B_{n-1}\zeta_n$, $n \geq 1$.
- $\mathcal{L}_n = span\{B_k\}_{k=0}^{n} = \left\{f = \frac{P}{\pi_n} : P \in \mathbb{P}_n\right\}$ with $\pi_n(z) = \prod_{j=1}^{n} 1 - \overline{\alpha_j}z$ and $\mathbb{P}_n = span\{1, z, \ldots, z^n\}$.
- Set $\omega_n(z) = \prod_{j=1}^{n}(z - \alpha_j)$. Thus, $B_n(z) = \frac{\omega_n(z)}{\pi_n(z)}$.
- Substar conjugate of f: $f_*(z) = \overline{f(1/\bar{z})}$. We can define: $\mathcal{L}_{n*} = \{f \ : f_* \in \mathcal{L}_n\} = \left\{f = \frac{Q}{\omega_n} : Q \in \mathbb{P}_n\right\}$.

Rational generalization of well known concepts

- For p, q non-negative integers, rational (multipoint) analog of the space of Laurent polynomials:

$$\begin{aligned} \mathcal{R}_{-p,q} &= \mathcal{L}_{p*} + \mathcal{L}_q = span\{B_k \ : k = -p, \dots, -1, 0, 1, \dots, q\} \\ &= \left\{ f = \frac{P}{\omega_p \pi_q} : \ P \in \mathbb{P}_{p+q} \right\}, \end{aligned}$$

  where $B_{-k}(z) = B_{k*}(z) = \frac{1}{B_k(z)}$.

  Functions in $\mathcal{R}_{-p,q}$ have their poles in $\{\alpha_k\}_{k=1}^{p} \cup \{1/\overline{\alpha_l}\}_{l=1}^{q}$.

- Super-star conjugation: $f^*(z) = B_n(z) f_*(z)$, for $f \in \mathcal{L}_n \backslash \mathcal{L}_{n-1}$.
- $\mathcal{L}_n(\beta) = \{f \in \mathcal{L}_n : \ f(\beta) = 0\}$.
- Blaschke products are useful for computations:
  - $B_{-k}(z) = B_{k*}(z) = \overline{B_k(z)}$ for $z \in \mathbb{T}$. $B_n^* \equiv 1$.
  - $f = p_n/\pi_n \in \mathcal{L}_n \backslash \mathcal{L}_{n-1}$, then $f_* = p_n^*/\pi_n^*$ and $f^* = p_n^*/\pi_n$.
  - ...

Positive rational quadrature formulas on the unit circle

- Aim: estimations of $I_{\mathring{\mu}}(f) = \int_{-\pi}^{\pi} f(z) d\mathring{\mu}(\theta)$, with $z = e^{i\theta}$ by $I_n(f) = \sum_{k=1}^{n} \lambda_k f(z_k)$.
- Problem: find nodes and weights such that $I_{\mathring{\mu}}(f) = I_n(f)$ for all $f \in \mathcal{R}_{-p(n),q(n)}$, with $p(n) + q(n)$ as large as possible.

  Basic requirement: rational moments $\mu_k = \int_{-\pi}^{\pi} \overline{B_k(z)} d\mathring{\mu}(\theta)$ with $z = e^{i\theta}$ must exist $\forall k$. Notice: $\mu_{-k} = \overline{\mu_k}$.
- Gram-Schmidt process to the basis $B_0, B_1, \ldots, B_n$:

  $\{\rho_j\}_{j=0}^{n}$ orthonormal basis for $\mathcal{L}_n$.

  Repeated for all n: orthonormal system $\{\rho_j\}_{j=0}^{\infty}$. ORF.

  $\rightsquigarrow$ not unique, we can always multiply with a unimodular constant. Normalization: choosing leading coefficient $\kappa_n = \overline{\rho_n^*(\alpha_n)}$ of ρ_n to be real and positive.

Positive rational quadrature formulas on the unit circle

Para-orthogonal rational functions, PORF.

As in the ordinary polynomial situation, set

$\chi_n(z,\tau_n) = C_n\left[\rho_n(z) + \tau_n\rho_n^*(z)\right]$, with $C_n \neq 0$ and $\tau_n \in \mathbb{T}$. Then:

- $\chi_n^*(z,\tau_n) = \kappa\chi_n(z,\tau_n)$, for $\kappa = \frac{\overline{C_n}}{C_n}\overline{\tau_n} \in \mathbb{T}$ and $z \in \mathbb{C}\backslash\{1/\overline{\alpha_1},\ldots,1/\overline{\alpha_n}\}$. (invariance)
- $\langle\chi_n, f\rangle_{\mathring{\mu}} = 0$ for all $f \in \mathcal{L}_{n-1}(\alpha_n)$ and $\langle\chi_n, 1\rangle_{\mathring{\mu}} \cdot \langle\chi_n, B_n\rangle_{\mathring{\mu}} \neq 0$.
- χ_n has exactly n distinct zeros on $\{z_j\}_{j=1}^n \subset \mathbb{T}$.
- Taking such zeros as nodes, there exist positive numbers $\{\lambda_j\}_{j=1}^n$ s.t. $I_n(f) = I_{\mathring{\mu}}(f)$, for all $f \in \mathcal{R}_{-(n-1),n-1}$.

Rational Szegő quadrature formula

Positive rational quadrature formulas on the unit circle

- Maximal domain of exactness since there cannot exist a positive n-point rule with nodes on $\mathbb{T}$ that is exact neither $\mathcal{R}_{-n,n-1}$ or $\mathcal{R}_{-(n-1),n}$.
- The weights $\{\lambda_k\}_{k=1}^{n}$ are positive and can be computed by $\lambda_k^{-1} = \sum_{j=0}^{n-1} |\rho_j(z_k)|^2 > 0, \; k = 1, \ldots, n$.

Rational Szegő recurrence

Rational Szegő recurrence:

$\rho_0(z) \equiv 1$ and for $n \geq 1$,

$$\begin{pmatrix} \rho_n(z) \\ \rho_n^*(z) \end{pmatrix} = M_n(z) \begin{pmatrix} \rho_{n-1}(z) \\ \rho_{n-1}^*(z) \end{pmatrix}$$

with

$$M_n(z) = e_n \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z} \begin{pmatrix} 1 & \delta_n \\ \overline{\delta_n} & 1 \end{pmatrix} \begin{pmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{pmatrix}$$

where

$$\delta_n = \frac{\rho_n(\alpha_{n-1})}{\rho_n^*(\alpha_{n-1})} = -\frac{\langle \frac{z-\alpha_{n-1}}{1-\overline{\alpha_n}z} \rho_{n-1}(z), \rho_k(z) \rangle_{\mathring{\mu}}}{\langle \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z} \rho_{n-1}^*(z), \rho_k(z) \rangle_{\mathring{\mu}}} \in \mathbb{D}, \quad \forall k = 0, 1, \ldots, n-1.$$

(δ_n: Rational Verblunsky coefficients) and

$$0 < e_n = \sqrt{\frac{1-|\alpha_n|^2}{1-|\alpha_{n-1}|^2} \frac{1}{1-|\delta_n|^2}}.$$

Para-orthogonal rational functions (PORF)

- PORF: by applying the rational Szegő recurrence for ρ_n and ρ_n^*, χ_n can be expressed in terms of ρ_{n-1} and ρ_{n-1}^*:

$$\begin{aligned} \chi_n(z,\tau_n) &= \rho_n(z) + \tau_n \rho_n^*(z) \\ &= C_n \frac{1-\overline{\alpha_{n-1}}z}{1-\overline{\alpha_n}z} \left[\zeta_{n-1}(z)\rho_{n-1}(z) + \tau_n^* \rho_{n-1}^*(z)\right] \end{aligned}$$

with $C_n \neq 0$ and $\tau_n^* \in \mathbb{T}$.

- Thus: it will be suffice to compute ORF of degree $n-1$. The nodes of the rational Szegő rule are the zeros of

$$\chi_n(z,\tau_n^*) = (z-\alpha_{n-1})\rho_{n-1}(z) + \tau_n^*(1-\overline{\alpha_{n-1}}z)\rho_{n-1}^*(z),$$

for some $\tau_n^* \in \mathbb{T}$.

Fixing nodes in advance in the quadrature formula

- Rational Szegő: always exist, depend on $\tau_n^* \in \mathbb{T}$ (arbitrary), positive and exact in $\mathcal{R}_{-(n-1),n-1}$.
- Rational Szegő-Radau: always exist, positive, exact in $\mathcal{R}_{-(n-1),n-1}$ and it is unique.

  To fix a node $u \in \mathbb{T}$, take $\tau_n^* = \frac{(\alpha_{n-1}-u)\rho_{n-1}(u)}{(1-\overline{\alpha_{n-1}}u)\rho_{n-1}^*(u)} \in \mathbb{T}$.

Fixing nodes in advance in the quadrature formula

- Rational Szegő-Lobatto: To fix two distinct nodes $u, v \in \mathbb{T}$.

  Rational extension of the polynomial situation (C. Jagels and L. Reichel, JCAM 2007).
  Modify the last rational Verblunsky parameter:
  $\delta_{n-1} \rightsquigarrow \tilde{\delta}_{n-1} \in \mathbb{D}$ and to take an appropriate parameter $\tilde{\tau}_n^*$ s.t. the n-th para-orthogonal rational function has u, v among it zeros.

  Problem: given $\mathring{\mu}$, $\{\alpha\} \in \mathbb{D}$ and $u, v \in \mathbb{T}$, to find a positive q.f. with u, v prescribed nodes and a minimal number of other nodes, mutually distinct, exact in $\mathcal{R}_{-(n-1),n-1}$.

  Three possible situations:

Fixing nodes in advance in the quadrature formula

Three possible situations:

1. u, v happen to be among the zeros of a para-orthogonal rational function of degree n. Exactness in $\mathcal{R}_{-(n-1),n-1}$.

2. u, v happen to be among the zeros of a para-orthogonal rational function of degree $n+1$. Exactness in $\mathcal{R}_{-n,n} \supset \mathcal{R}_{-(n-1),n-1}$.

Fixing nodes in advance in the quadrature formula

Three possible situations:

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2. u, v happen to be among the zeros of a para-orthogonal rational function of degree $n+1$. Exactness in $\mathcal{R}_{-n,n} \supset \mathcal{R}_{-(n-1),n-1}$.

Fixing nodes in advance in the quadrature formula

Three possible situations:

3 Generic case: there exist a proper $(n+1)$-point rational Szegő-Lobatto rule. Exactness in $\mathcal{R}_{-(n-1),n-1}$.

Set:

$$\xi_1 = \zeta_n(u) \in \mathbb{T}, \qquad \xi_2 = \zeta_n(v) \in \mathbb{T},$$
$$\tau_1 = -\frac{\zeta_{n-1}(u)\rho_{n-1}(u)}{\rho^*_{n-1}(u)} \in \mathbb{T}, \quad \tau_2 = -\frac{\zeta_{n-1}(v)\rho_{n-1}(v)}{\rho^*_{n-1}(v)} \in \mathbb{T}.$$

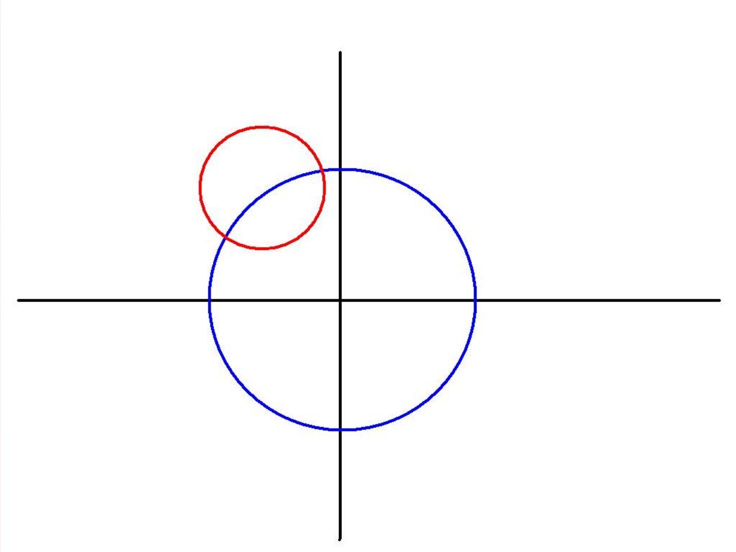
$\rightsquigarrow$ Modified $\delta_n \to \tilde{\delta}_n \in \mathcal{C} \cap \mathbb{D} \neq \varnothing$, with $\mathcal{C}$ the circle with center $c = \frac{\xi_1 - \xi_2}{\overline{\tau_1}\xi_1 - \overline{\tau_2}\xi_2}$ and radius $r = |1 - c\overline{\tau_2}|$.

$\rightsquigarrow$ It holds $\mathcal{C} \cap \mathbb{T} = \{\tau_1, \tau_2\}$.

$\rightsquigarrow$ If the center and the radius are infinite, then the circle becomes a straight line.

$\rightsquigarrow$ Particular choice of $\tilde{\tau}^*_{n+1} = \xi_1 \frac{\tau_1 - \tilde{\delta}_n}{1 - \tau_1 \overline{\tilde{\delta}_n}}$.

Fixing nodes in advance in the quadrature formula



Some references

- A. Bultheel, R. Cruz-Barroso, K. Deckers and P. González-Vera.- **Rational Szegő quadratures associated with Chebyshev weight functions**, *Math. Comp. vol 78, No. 266 (2009), 1031–1059.*
- *A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad.*- **Rational quadrature formulas on the unit circle with prescribed nodes and maximmal domain of validity**, IMA J. Numer. Anal. 30 (2010), 940–963.
- A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad.- **Computation of rational Szegő-Lobatto quadrature formulas**, *Appl. Num. Math. 60 (2010), 1251–1263.*
- *K. Deckers, R. Cruz-Barroso and F. Perdomo-Pío.*- **Positive rational interpolatory quadrature formulas on the unit circle and the interval**, Appl. Num. Math. 60 (2010), 1286–1299.

A first new result: three prescribed nodes

- General open problem (hard!): to characterize a polynomial and rational quadrature formula on the unit circle with an arbitrary number of prescribed nodes and maximal domain of exactness.
- Next step: to fix three nodes.
- The solution now is completely different!: the rule may not exist.
- A characterization can be obtained: a positive quadrature formula exist under conditions on $\mathring{\mu}$ and $\{\alpha\} \subset \mathbb{D}$.
- Idea: in the rational Szegő-Lobatto rule, to fix the free rational Verblunsky parameter $\tilde{\delta}_n$ on $\mathcal{C} \cap \mathbb{D}$ s.t. the PORF $\chi_{n+1}(z, \tilde{\tau}^*_{n+1})$ has u, v, w among its zeros.

A first new result: three prescribed nodes

- Theorem: this $(n+1)$-point rule exists (positive, unique and exact in $\mathcal{R}_{-(n-1),n-1}$), iff

  $\tilde{\delta}_n =$(long expression depending only on $u, v, w, \mathring{\mu}, \{\alpha\}) \in \mathcal{C}$.

  Problem!: this parameter does not always satisfy $|\tilde{\delta}_n| < 1$.

- Idea of the proof:

  $\rightsquigarrow$ Rational Szegő recurrence: $\tilde{\chi}_{n+1}(z, \tilde{\tau}^*_{n+1})$ is written in terms of $\{\rho_{n-1}, \rho^*_{n-1}, \tilde{\delta}_n, \tilde{\tau}^*_{n+1}\}$.

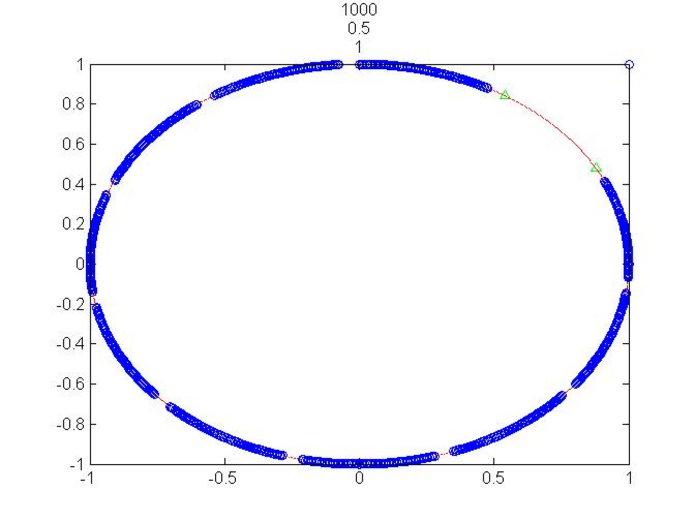
  $\rightsquigarrow$ Construct Rational Szegő-Lobatto rule to fix u and $v \rightarrow$ $\tilde{\delta}_n \in \mathcal{C} \cap \mathbb{D}$ (free) and $\tilde{\tau}^*_{n+1}$ uniquely determined from $\tilde{\delta}_n$.

  $\rightsquigarrow$ To introduce the expression for $\tilde{\tau}^*_{n+1}$ + expressions for the center and radius + $\tilde{\delta}_n \in \mathcal{C} \cap \mathbb{D}$ + to fix w + tedious calculations (and some luck!) $\rightarrow$ explicit expression for $\tilde{\delta}_n$.

An example

- $\mathring{\mu}$: Lebesgue measure on the unit circle.
- ORF: Takenaka-Malmquist basis, $\{1\} \cup \{\frac{(1-|\alpha_{k+1}|)^{1/2} z}{1-\overline{\alpha_{k+1}} z} B_k(z)\}_{k=0}^{n-1}$.
- Two nodes u and v prescribed (green triangles). A third node w moving around the unit circle.
- Take $n = 10$.
- Blue region: accepted rational quadrature rule (existence and positive).

An example



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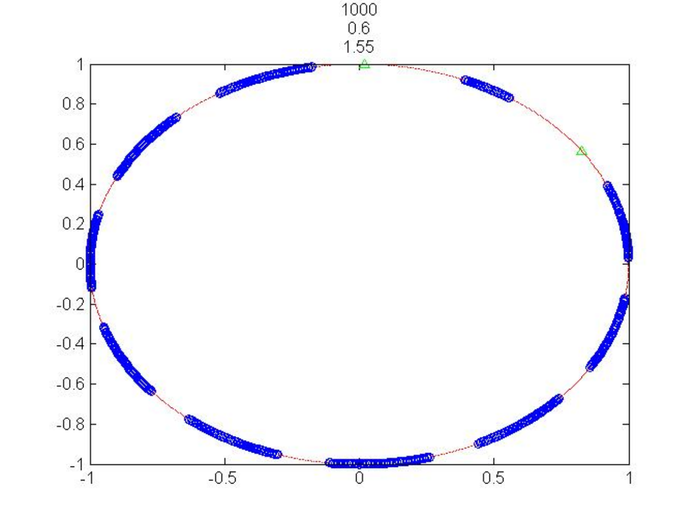

A second new result (in progress)

- Joukowsky transform: $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $x = J(z)$. Maps $\mathbb{D}$ onto the cut Riemann sphere $\overline{\mathbb{C}} \backslash [-1,1]$ and $\mathbb{T}$ onto $[-1,1]$.
- Connection between quadrature formulas on the interval and the unit circle:
  - Polynomial situation: A. Bultheel, L. Daruis and P. González-Vera, JCAM, 2001.
  - Rational extension: A. Bultheel, R. Cruz-Barroso, K. Deckers and F. Perdomo-Pío, Appl. Num. Math., 2010.
- Gaussian quadrature formulas on the interval with a prescribed node anywhere:
  - Polynomial situation: A. Bultheel, R. Cruz-Barroso and M. Van-Barel, Calcolo, 2010.
  - Rational extension: A. Bultheel, K. Deckers and F. Perdomo-Pío, Jaén J. Approx., 2011.

A second new result (in progress)

- Next step: quadrature formulas on the interval with a prescribed node inside the interval of integration and possibly one or both endpoints.
  - Polynomial situation: A. Bultheel, R. Cruz-Barroso and M. Van-Barel, Calcolo, 2010.
  - Rational extension: A first approach has been done by K. Deckers (in progress, not submitted yet!), by considering quasi-orthogonal rational functions on the interval: $\phi_n(z) = \rho_n(z) + a\rho_{n-1}(z)$.
- Alternative approach (in progress): to use Joukowsky transform and symmetric rational Szegő-Lobatto rules.

A second new result (in progress)

- Theorem: Suppose that $\{\alpha_1, \ldots, \alpha_{n-1}\}$ are real or appear in complex conjugate pairs and that $\mathring{\mu}$ is symmetric. Set

$$\upsilon_n = \prod_{j=1}^{n} \eta_j \in \{\pm 1\} \quad \text{with} \quad \eta_j = \begin{cases} -\frac{\overline{\alpha}_j}{|\alpha_j|} & \textit{if} \quad \alpha_j \neq 0, \\ 1 & \textit{if} \quad \alpha_j = 0. \end{cases}$$

Then,

1. The zeros of $\chi_n(z, \tau_n)$ appear in complex conjugate pairs iff $\tau_n = \pm 1$.
2. $\chi_n(z, \tau_n)$ has a zero in
   1. $z = 1$ iff $\tau_n = -\upsilon_n$,
   2. $z = -1$ iff $\tau_n = (-1)^{n+1}\upsilon_n$.
3. In a rational Szegő rule, the weights associated to two complex conjugate nodes are equal, iff, $\tau_n = \pm 1$.

A second new result (in progress)

- By considering symmetric rational Szegő-Lobatto quadrature formulas (u and $\bar{u}$ prescribed) and connecting with the interval by Joukowsky transform $\rightsquigarrow$
  Characterization of rational Gaussian quadrature formulas with a node prescribed inside and possibly one or both endpoints.

- Computation:
  - Rational Gaussian formulas: generalized eigenvalue problem (A. Bultheel and J. Van Deun, Numer. Algor., 2007)
  - Rational Szegő formulas:
    Operator Mőbius transformations of Hessenberg or CMV matrices (L. Velázquez, J. Funct. Anal., 2008) and (A. Bultheel and M.J. Cantero, Numer. Algor. 2009).

A second new result (in progress)

- An algorithm for the computation of Gaussian rules with a prescribed node by passing to the unit circle in the polynomial situation is presented in the PhD Thesis of F. Perdomo-Pío (2013): key fact:
  - Geronimus relations (computation of Jacobi matrices from Verblunsky coefficients for two measures related by the Joukowsky transform)
  - A result due to L. Garza, J. Hernández and F. Marcellán (Numer. Algor, 2008): algorithm for the same connection in the opposite direction by using LU decomposition.
- Extension to the rational case: ?

A second new result (in progress)

Thank You!