Interpolation and meromorphic extension

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Based on the joint work

Meromorphic continuation of functions and arbitrary distribution of interpolation points

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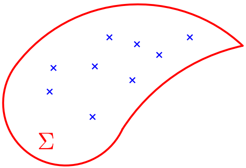
Summary

1. Definitions and review
2. Basic ideas
3. Our results
4. Open problems

On the interpolation table

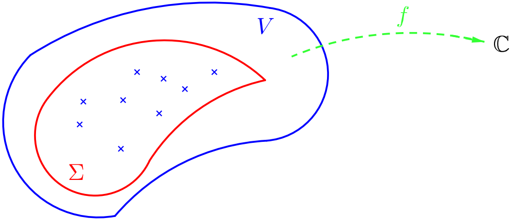
Let $\Sigma \subset \mathbb{C}$ be a compact set with connected complement



$$\{\zeta_{j,n} : j = 1, \ldots, n; n = 1, \ldots\} \subset \Sigma$$

$$\begin{array}{l} \zeta_{1,1} \\ \zeta_{2,1}, \zeta_{2,2} \\ \ldots\ldots \\ \zeta_{n,1}, \zeta_{n,2}, \ldots, \zeta_{n,n} \\ \vdots \end{array}$$

$$w_n(z) = \prod_{j=1}^{n} (z - \zeta_{j,n}), \quad n = 1, 2, \ldots$$



Let $f \in H(V)$, V be an open set, $V \supset \Sigma$, $n \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{\geq 0}$, there exist polynomials P y Q such that:

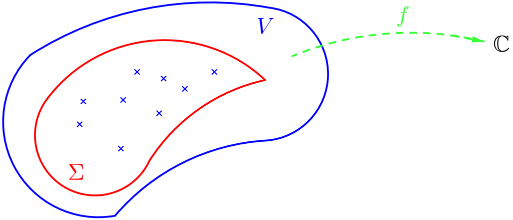
- $\deg(P) \leq n$, $\deg(Q) \leq m$, $Q \not\equiv 0$.
-

$$(Q\,f - P)/w_{n+m+1} \in \mathcal{H}(V)$$

Any pair of such polynomials P, Q defines a unique rational function

$$\Pi_{n,m} = P/Q \text{ called multi-point Padé approximant}$$

of type (n, m) for f.



Let $f \in H(V)$, V be an open set, $V \supset \Sigma$, $n \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{\geq 0}$, there exist polynomials P y Q such that:

- $\deg(P) \leq n$, $\deg(Q) \leq m$, $Q \not\equiv 0$.
-

$$(Q\,f - P)/w_{n+m+1} \in \mathcal{H}(V) \Rightarrow (Q\,f - P)(\zeta_{j,n+m+1}) = 0$$

Any pair of such polynomials P, Q defines a unique rational function

$$\Pi_{n,m} = P/Q \text{ called multi-point Padé approximant}$$

of type (n, m) for f.

Multi-point interpolation. Particular cases

$$\frac{Qf - P}{w_{n+m+1}} \in \mathcal{H}(V), \quad \Pi_{n,m} = P/Q.$$

- $m = 0$, $w_{n+1}(z) = (z - z_0)^{n+1}$, $\Pi_{n,0}$ Taylor polynomials.
- $m = 0$, $\zeta_{j,n+1} \neq \zeta_{k,n+1}$, if $j \neq k$, $\Pi_{n,0}$ Lagrange interpolation.
- $m = 0$, $w_{n+1}(z)/w_n(z) = (z - \zeta_{n+1,n+1})$ Jacobi series.
- $w_{n+m+1}(z) = (z - z_0)^{n+m+1}$, $\Pi_{n,m}$ Padé approximants.

Best approximation $\Rightarrow$ interpolation

- Uniform aproximation. If r^*/s^* satisfies

  $$\|f - r^*/s^*\|_{[a,b]} = \min\{\|f - r/s\|_{[a,b]} : \deg(r) \le n, \deg(s) \le m\}$$

  then (Chebychev Alternation Theorem)
  $\exists \zeta_j \in [a, b]$, $j = 0, \ldots, n + m$ such that

  $$f(\zeta_j) - r^*(\zeta_j)/s^*(\zeta_j) = 0$$

  where $[a, b] \subset \mathbb{R}$ and $\| \cdot \|_{[a,b]}$ is the uniform norm.
- Fourier series. If μ is a nontrivial positive measure with finite moments on an interval $[a, b] \subset \mathbb{R}$, and s_n is the $n-$th partial sum of a function $f \in \mathcal{C}([a, b])$ in $L^2(\mu)$, then (orthogonality conditions) $\exists \zeta_j \in [a, b]$, $j = 0, \ldots, n$ such that

  $$f(\zeta_j) - s_n(\zeta_j) = 0, \quad j = 0, 1, \ldots, n.$$

Interpolation and orthogonality

$$(Q\,f - P)/w_{n+m+1} \in \mathcal{H}(V)$$

$\Downarrow$ *Cauchy Theorem*

$$\int_\Gamma \frac{Q(\zeta)f(\zeta) - P(\zeta)}{w_{n+m+1}(\zeta)}\zeta^k d\zeta = 0,$$

where Γ is a closed curve in V and $k \in \mathbb{Z}_{\geq 0}$. If $\Sigma \subset \mathrm{int}(\Gamma)$, $0 \leq k \leq m-1$,

$\Downarrow$ *Cauchy Theorem*

$$\int_\Gamma \zeta^k Q(\zeta)\frac{f(\zeta)d\zeta}{w_{n+m+1}(\zeta)} = 0$$

Taylor Polynomials, $w_n(z) = z^n$, $m = 0$

Theorem (Cauchy-Abel). Let f be an analytic function on a neighborhood of 0.

- Let f be analytic in $D(0, R_0)$ but not on $\overline{D(0, R_0)}$. Then $\forall z \in D(0, R_0)$,

$$\limsup_{n\to\infty} |f(z) - \Pi_{n,0}(z)|^{1/n} \leq \frac{|z|}{R_0} < 1.$$

- Let $z \neq 0$,

$$\limsup_{n\to\infty} |f(z) - \Pi_{n,0}(z)|^{1/n} = \frac{|z|}{R} < 1,$$

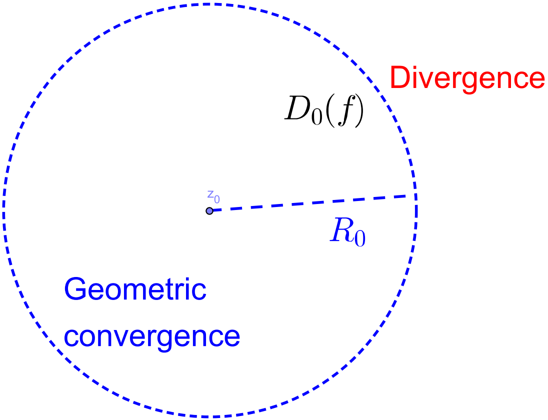
then $R = R_0$; i.e., f has an analytic extension to $\{|z| < R\}$.

- For $|z| > R_0$, it holds

$$\limsup_{n\to\infty} |\Pi_{n,0}(z)|^{1/n} = \frac{|z|}{R_0};$$

in particular, $\{\Pi_n(z)\}$ diverges in $\overline{D(0, R_0)}^c$.

$$w_n(z) = z^n,$$
$$\limsup_{n\to\infty} |f(z) - \Pi_{n,0}(z)|^{1/n} = \frac{|z|}{R} < 1,\ z \neq 0 \Rightarrow R = R_0,$$
$$\limsup_{n\to\infty} |\Pi_{n,0}(z)|^{1/n} = \frac{|z|}{R_0} > 1$$



Multi-point analogous results?

Role of logarithmic potential, $m = 0$

$$\Pi_{n,0}(z) = \frac{1}{2\pi i}\int_\Gamma \frac{w_{n+1}(\zeta) - w_{n+1}(z)}{\zeta - z}\frac{f(\zeta)\,d\zeta}{w_{n+1}(\zeta)},$$

Γ a closed part $\Sigma \subset \mathrm{int}(\Gamma)$, $\Gamma \subset V$. We have

$$f(z) - \Pi_{n,0}(z) = \frac{w_{n+1}(z)}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\frac{d\zeta}{w_{n+1}(\zeta)}, \quad z \in \mathrm{int}(\Gamma).$$

$$\Rightarrow |f(z) - \Pi_{n,0}(z)| \leq C\frac{|w_{n+1}(z)|}{\min_{\zeta\in\Gamma}|w_{n+1}(\zeta)|}$$

Role of logarithmic potential:

$$|w_{n+1}(z)|^{1/n} = e^{-\frac{n+1}{n}P_{\mu_{w_{n+1}}}(z)} := e^{\frac{n+1}{n}\int \log|z-t|d\mu_{w_{n+1}}(t)} \underset{n\to\infty}{\longrightarrow} ?$$

$$P_{\mu_{w_{n+1}}}(z) := -\int \log|z-t|d\mu_{w_{n+1}}(t),$$

where $\mu_{w_{n+1}}(A) := \frac{1}{n+1}\sum_{\zeta : w_{n+1}(\zeta)=0}\delta_\zeta(A)$, A boreliano en $\mathbb{C}$.

Potential theory

Let μ, $\{\mu_n\}$ be a probability measures with compact support $K \subset \mathbb{C}$.

- Logarithmic potential of μ:

$$P_\mu(z) = P(\mu; z) = -\int_K \log|z-\zeta|\, d\mu(\zeta)$$

- Energy of μ: $E(\mu) = \int_K P(\mu; z)\, d\mu(z)$

- Minimal energy on K: $E(K) = \inf_\mu E(\mu)$

- Logarithmic potential of K: $\operatorname{cap}(K) = \exp\{-E(K)\}$

- Equilibrium measure of K: μ_K if $E(K) = E(\mu_K)$ $(\operatorname{cap}(K) > 0)$

- Weak$*$ convergence:

$$*-\lim_n \mu_n = \mu \overset{\text{def}}{\Leftrightarrow} \lim_n \int g(t)\, d\mu_n(t) = \int g(t)\, d\mu(t), \quad \forall g \in \mathcal{C}(K),$$

taking $\overset{g(t)=\log|z-t|}{\Leftrightarrow}$ $\lim_n P(\mu_n, z) = P(\mu, z),\ z \in \mathbb{C} \setminus K.$

- Decent principle: Moreover, if $\lim_n z_n = z_0$,

$$\lim_{n\to\infty} P(\mu_n, z_n) \geq P(\mu, z_0).$$

Limit distribution $|w_{n+1}(z)|^{1/n} = e^{-\frac{n+1}{n}P(\mu_{w_{n+1}},z)}$

$$*-\lim_n \mu_{w_{n+1}} = \mu \overset{\text{def}}{\Leftrightarrow} \lim_n \int g(t)\, d\mu_{n+1}(t) = \int g(t)\, d\mu(t), \quad \forall g \in \mathcal{C}(\Sigma),$$

$$\overset{g(t)=\log|z-t|}{\Leftrightarrow} \lim_n |w_{n+1}(z)|^{1/n} = e^{-P(\mu,z)},\ z \in \mathbb{C} \setminus \Sigma.$$

- $\Sigma = \{0\}$, $w_n(z) = z^n$, Taylor polynomials,

$$\mu = \delta_{\{0\}}.$$

- $\Sigma = [-1,1]$, $w_n(z) = cos(n \arccos z)$, $z \in [-1,1]$, Lagrange interpolation on the zeros of Chebyshev polynomials,

$$d\mu(t) = \frac{1}{\pi}\frac{dt}{\sqrt{1-t^2}}.$$

- $\Sigma = [-1,1]$, the zeros of w_n are uniformly distributed on $[-1,1]$, Lagrange interpolation on,

$$d\mu(t) = \frac{dt}{2}.$$

Runge's phenomenon, $f(z) = \frac{1}{1+a^2z^2}$

Runge's phenomenon [1901]. Interpolation on uniform distributed points $[-1, 1]$,

$$f(z) = \frac{1}{1 + a^2 z^2}, \quad a > 1.$$

There exists $a^* \in (0, 1)$ such that

$$\lim_{n\to\infty} \Pi_{n,0}(z) = f(z), \quad z \in (-a^*, a^*),$$

$$\{\Pi_{n,0}(z)\} \quad \text{diverges } z \in ((-1, a^*) \cup (a^*, 1)).$$

$$|f(z) - \Pi_{n,0}(z)| \leq C \frac{|w_{n+1}(z)|}{\min_{\zeta\in\Gamma} |w_{n+1}(\zeta)|}$$

$\Gamma \subset V$

Runge's phenomenon. $f(z) = \frac{1}{1+a^2z^2}$. Cont.

Let $R_{\mu,0}$ be the largest R such that f has an analytic extension to $\{z : e^{-P(\mu,z)} < R\}$

$$D_{\mu,0} = \{z : e^{-P(\mu,z)} < e^{-P(\mu,i/a)}\}, \quad R_{\mu,0} = e^{-P(\mu,i/a)}.$$

Theorem

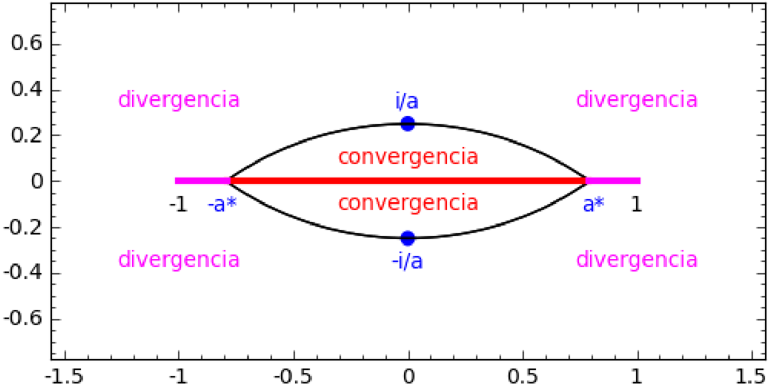
Assume that f is an analytic function on $[-1,1]$.

- *We have*

$$\lim_{n\to\infty} |f(z) - \Pi_{n,0}(z)|^{1/n} \le \frac{e^{-P(\mu,z)}}{R_{\mu,0}} < 1,$$

  uniformly on compact subset of $D_{\mu,0}$.
- *If f has a pole in z^*, $f \in \mathcal{H}(\{z : e^{-P(\mu,z)} < e^{-P(\mu,z^*)+\epsilon}\} \setminus \{z^*\})$, then $\{\Pi_{n,0}(z)\}$ diverges in $\{z : e^{-P(\mu,z^*)} < e^{-P(\mu,z)}\}$.*

Runge's phenomenon. $f(z) = \frac{1}{1+a^2z^2}$. Cont.



$$D_{\mu,0} = \{z : e^{-P(\mu,z)} < e^{-P(\mu,i/a)}\}, \quad R_{\mu,0} = e^{-P(\mu,i/a)}.$$

- Walsh [1935], $m = 0$ fixed, general interpolation table. Convergence in $D_{\mu,0}$ and divergence result for particular interpolation table (roots of the unit).
- Kakehashi [1955], $m = 0$ fixed, divergence in $\overline{D_{\mu,0}}^{c}$ for particular interpolacin table.
- Saff [1972], $m \geq 0$ fixed, convergence of $\{\Pi_{n,m}\}_{n\in\mathbb{N}}$ in $D^*_{\mu,m}$ for general interpolation table.
- Vavilov [1976], $m \geq 0$ fixed, inverse-type theorem for Padé interpolation.
- Wallin [1984], $m \geq 0$ fixed, type-Runge theorem for funciones meromorfas and general interpolation table.
- Grothmann [1996], $m = 0$ fixed, inverse theorem for extremal interpolation table.
- Khistoforov [2008], $m = 0$ fixed, a Jentzsch-Szegő-type theorem a, Hadamard-type theorem for Kakehashi interpolation table.

Our results

Let $R_{\mu,m}$ be the largest R such that f has a meromorphic extension to $\{z : e^{-P(\mu,z)} < R\}$ with at most m poles counting multiplicities.

$$D_{\mu,m} := \{z : e^{-P(\mu,z)} < R_{\mu,m}\}.$$

We characterize this region in terms of the behavior of $\{\Pi_{n,m}\}_n$.

Hausdorff contents

Let $A \subset \mathbb{C}$ and $\mathcal{U}(A)$ be the class of all coverings of A by a denumerable set of disks. Set

$$\sigma(A) = \inf \left\{ \sum_{i \in I} |U_i| \, : \, \{U_i\}_{i \in I} \in \mathcal{U}(A) \right\},$$

where $|U_i|$ is the radius of U_i. $\sigma(A)$ denotes the 1−dimensional Hausdorff contents of A. This function is an exterior Caratheodory measure; so it is monotone and $\sigma-$subaditive.

$\sigma-$content converge: $\{\varphi_n\}$ converges in $\sigma-$ content to φ on compact subsets of D if $\forall \epsilon > 0$, $\forall K \subset D$, K compact, we have

$$\lim_{n\to\infty} \sigma\left(\{z \in K : |\varphi_n(z) - \varphi(z)| > \varepsilon\}\right) = 0.$$

Divergence results

Theorem

Let the measure μ be the asymptotic zero distribution of the sequence of interpolation points given by $\{w_n\}_{n\in\mathbb{N}}$. Suppose that the sequence $\{\Pi_{n,m}\}_{n\geq m}$ converges in σ-content on compact subsets of a neighborhood of the point $z_0 \in \mathbb{C}\setminus\Sigma$. Then, $z_0 \in D_{\mu,m}$.

Corollary

From above theorem and Osgood-Caratheodory's Theorem it follows that if the sequence $\{\Pi_{n,m}\}_{n\geq m}$ converges pointwise on a neighborhood of the point $z_0 \in \mathbb{C}\setminus\Sigma$, then $z_0 \in D_{\mu,m}$. So, $\{\Pi_{n,m}\}_{n\in\mathbb{N}}$ diverges in a dense subset of $\overline{D_{\mu,m}}^c$.

Inverse-type Theorem

Set

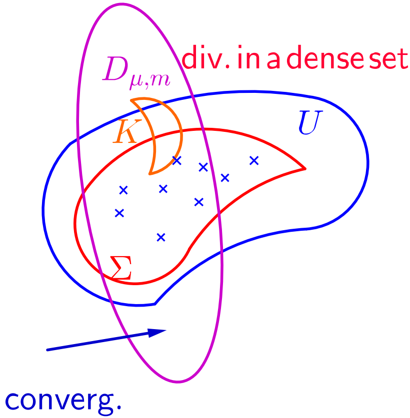
$$\rho_\mu(K) := \sup\{e^{-P(\mu;z)} : z \in K\} := \|e^{-P(\mu;\cdot)}\|_K.$$

Theorem

Let the measure μ be the asymptotic zero distribution of the sequence of interpolation points given by $\{w_n\}_{n\in\mathbb{N}}$. Let K be a regular compact set for which the value $\rho_\mu(K)$ is attained at a point that does not belong to the interior of Σ. Suppose that the function f is defined on K and fulfills

$$\limsup_{n\to\infty} \|f - \Pi_{n,m}\|_K^{1/n} \leq \frac{\rho_\mu(K)}{R} < 1. \tag{1}$$

Then, $R_{\mu,m} \geq R$, that is, f admits meromorphic continuation with at most m poles on the set $\{z : e^{-P(\mu,z)} < R\}$.



$$\limsup_{n\to\infty} \|f - \Pi_{n,m}\|_K^{1/n} \leq \frac{\rho_\mu(K)}{R} < 1.$$

Then, $R_{\mu,m} \geq R$, i.e. f has a meromorphic continuation to $\{z : e^{-V_\mu(z)} < R\}$ with at most m poles.

Some open problems

- Inverse problem: A Hadamard formula for $R_{\mu,0}$. Gonchar's conjecture

$$\frac{1}{R_{\mu,0}} = \limsup_{n\to\infty} \left| \int_\Gamma \frac{f(\zeta)}{w_{n+1}(\zeta)} d\zeta \right|^{1/n}$$

  where Γ is an arbitrary closed part such that $\Sigma \subset \text{int}(\Gamma)$ and $f \in \mathcal{H}(\overline{\text{int}(\Gamma)})$.
  Buslaev (2006), Kristoforov (2008).

- Direct problems:
  - Divergence: Prove or disprove the divergence of $\Pi_{n,0}$ in $\overline{D_{\mu,0}}^c$.
  - Quantitative results. Estimate the rate of convergence of $f - \Pi_{n,m}$ when f has singularity at Σ.
    Rakhmanov (1984). López Lagomasino-Martínez Finkelshtein (1995)

Thank you!