Spherical Functions and Orthogonal Polynomials

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Spherical Harmonics

A function $f = f(x, y, z)$ is harmonic if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

It is homogeneous of degree n if

$$f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z).$$

The homogeneous harmonic polynomials can be considered as functions on the unit sphere S^2 in R^3. They are called spherical harmonics.

Let (r, θ, ϕ) be ordinary polar coordinates in R^3:

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta.$$

In terms of these coordinates the Riemannian structure of R^3 is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 (\sin\theta)^2 d\phi^2,$$

and the Laplace operator is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2(\sin\theta)^2}\frac{\partial^2}{\partial \phi^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cos\theta}{r^2 \sin\theta}\frac{\partial}{\partial \theta}.$$

If $f = f(\theta)$ is a spherical harmonic of degree ℓ, then

$$\frac{d^2 f}{d\,\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{df}{d\theta} + \ell(\ell+1)f = 0.$$

By making the change of variables $y = (1+\cos\theta)/2$ we get

$$y(1-y)\frac{d^2 f}{d\,y^2} + (1-2y)\frac{df}{dy} + \ell(\ell+1)f = 0.$$

The bounded solution at $y = 0$, up to a constant, is ${}_2F_1(-\ell, \ell+1, 1; y)$. Since the Legendre polynomial of degree ℓ is given by

$$P_\ell(x) = {}_2F_1\left(\begin{matrix}-\ell\,,\,\ell+1\\1\end{matrix}; (1+x)/2\right),$$

we get that $f(\theta) = P_\ell(\cos\theta)f(0)$.

Let $o = (0, 0, 1)$ be the north pole of S^2, and let $\phi(p) = P_\ell(\cos(\theta(p)))$ for $p \in S^2$. Then ϕ is the unique spherical harmonic of degree ℓ, constant along the parallels and such that $\phi(o) = 1$.

The set V_ℓ of all complex linear combinations of translates $\phi_g(p) = \phi(g \cdot p)$, $g \in SO(3)$, is the linear space of all spherical harmonics of degree ℓ.

The action of $SO(3)$ in V_ℓ is an irreducible representation of $SO(3)$ of dimension $2\ell + 1$, and these are all.

Moreover we have the following unitary direct sum

$$L^2(S^2) = \sum_\ell V_\ell.$$

Legendre and Laplace found that the Legendre polynomials satisfy the following addition formula

$$P_\ell(\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\phi)$$
$$=P_\ell(\cos\alpha)P_n(\cos\beta)+2\sum_{k=1}^{\ell}\frac{(\ell-k)!}{(\ell+k)!}P_\ell^k(\cos\alpha)P_\ell^k(\cos\beta)\cos k\phi, \tag{1}$$

where the P_ℓ^k's are the associated Legendre polynomials.

By integrating (1) we get

$$P_\ell(\cos\alpha)P_\ell(\cos\beta)=\frac{1}{2\pi}\int_0^{2\pi}P_\ell(\cos\alpha\cos\beta+\sin\alpha\sin\beta\cos\phi)d\phi. \tag{2}$$

Moreover the Legendre polynomials can be determined as solutions to (2). This integral equation can now be expressed in terms of the function ϕ

on $\mathrm{SO}(3)$ defined by $\phi(g) = \phi(g \cdot o) = P_\ell(\cos(d(o, g \cdot o)))$. In fact (2) is equivalent to

$$\phi(g)\phi(h) = \int_K \phi(gkh)\, dk, \tag{3}$$

where K denotes the compact subgroup of $\mathrm{SO}(3)$ of all elements which fix the north pole o, and dk denotes the normalized Haar measure of K.

In fact, let A denote the subgroup of all elements of $\mathrm{SO}(3)$ which fix the point $(0, 1, 0)$. Then $\mathrm{SO}(3) = KAK$. Thus to prove () it is enough to consider rotations g and h around the y-axis through the angles α and β, respectively. Then if k denotes the rotation of angle ϕ around the z-axis we have

$gkh \cdot o =$

$(-\cos\alpha\cos\phi\sin\beta+\sin\alpha\cos\beta, -\sin\phi\sin\beta, \sin\alpha\cos\phi\sin\beta+\cos\alpha\cos\beta)^t.$

Thus $\cos(\theta(g \cdot o)) = cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\phi$ and

$$\phi(gkh) = P_\ell(cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\phi).$$

Then equation (3) becomes (2).

We have

$$\int_{S^2} \phi_m(p)\phi_n(p)\,dp = 0$$

$dp = \sin\theta\,d\theta\,d\phi$. If $m \neq n$, then

$$\int_0^{2\pi}\int_0^{\pi} P_m(\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta\,d\phi = 0$$

$$\int_0^{\pi} P_m(\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta = \int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$$

Lagrange

$$(2\ell+1)xp_\ell(x) = (\ell+1)p_{\ell+1}(x) + \ell p_{\ell-1}(x)$$

$$p_{-1}(x) = 0, \qquad p_0(x) = 1$$

The three term recursion relation revisited:

$\rho : SU(2) \to SO(3)$ covering isomorphism

V_ℓ becomes a $SU(2)$-module. It has a unique orthonormal basis $\{v_i\}_0^{2\ell}$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

$$Hv_i = 2(\ell - i)v_i, \quad Ev_i = iv_{i-1}, \quad Fv_i = (2\ell - i)v_{i+1}.$$

Up to a constant $\phi_\ell = v_\ell$.

$$V_\ell \otimes V_1 = V_{\ell+1} \oplus V_\ell \oplus V_{\ell-1}, \quad v_\ell \otimes v_1 = a_1 v_{\ell+1}^{\ell+1} + a_2 v_{\ell-1}^{\ell-1},$$

Clebsch-Gordan coefficients

$$|a_1|^2 = \frac{\ell+1}{2\ell+1}, \quad |a_2|^2 = \frac{\ell}{2\ell+1}.$$

Now

$$\phi_\ell(g) = \langle g \cdot v_\ell, v_\ell \rangle$$

In fact $\phi(g) = \langle g \cdot v_\ell, v_\ell \rangle \in V_\ell$ by Schur's orthogonality relations, it is K-invariant and $\phi(e) = 1$.

$$\langle g \cdot v_\ell, v_\ell \rangle \langle g \cdot v_1, v_1 \rangle = \langle g \cdot (v_\ell \otimes v_1), v_\ell \otimes v_1 \rangle,$$

$$\begin{aligned} \phi_\ell(g)\phi_1(g) &= \langle g \cdot (a_1 v_{\ell+1}^{\ell+1} + a_2 v_{\ell-1}^{\ell-1}), a_1 v_{\ell+1}^{\ell+1} + a_2 v_{\ell-1}^{\ell-1} \rangle \\ &= |a_1|^2 \phi_{\ell+1} + |a_2|^2 \phi_{\ell-1}, \end{aligned}$$

which is equivalent to

$$(2\ell + 1) x p_\ell(x) = (\ell + 1) p_{\ell+1}(x) + \ell p_{\ell-1}(x).$$

Zonal Spherical Functions

Elie Cartan and Herman Weyl: let G be a locally compact group and K a compact subgroup.

$$\phi(g)\phi(h) = \int_K \phi(gkh)\,dk.$$

Compact Two Point Homogeneous Manifolds

$$\begin{gathered} S^k = SO(k+1)/SO(k) \\ P^k(R) = SO(k+1)/O(k) \\ P^k(C) = SU(k+1)/U(k) \\ P^k(H) = Sp(k+1)/Sp(k) \times Sp(1) \\ P^2(Cay) = F_4/Spin(9) \end{gathered}$$

S^k: $p_n^{(\alpha,\alpha)}(cos\,\theta)$ $\alpha = \frac{k-2}{2}$

$P^k(R)$: $p_n^{(\alpha,\beta)}(cos\,\theta)$ $(\alpha,\beta) = (\frac{k-2}{2}, -\frac{1}{2})$

$P^k(C)$: $p_n^{(\alpha,\beta)}(cos\,\theta)$ $(\alpha,\beta) = (k-1, 0)$

$P^k(H)$: $p_n^{(\alpha,\beta)}(cos\,\theta)$ $(\alpha,\beta) = (2k-1, 1)$

$P^2(Cay)$: $p_n^{(\alpha,\beta)}(cos\,\theta)$ $(\alpha,\beta) = (7, 3)$

R^k: Bessel functions

hyperbolic spaces: Jacobi functions

Discrete two point homogeneous spaces

$\mathrm{S}_k \times \mathrm{Z}_2^k/\mathrm{S}_k$: Krawtchouk polynomials

$\mathrm{S}_\ell/\mathrm{S}_k \times \mathrm{S}_{\ell-k}$: Hahn polynomials

Spherical Functions

G locally compact group, K compact subgroup, $\delta \in \hat{K}$

$\Phi : G \longrightarrow \mathrm{End}(V)$ continuous such that $\Phi(e) = I$ and

$$\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky)\,dk$$

Lemma

(i) $\Phi(kgk') = \Phi(k)\Phi(g)\Phi(k')$

(ii) $\pi = \Phi_{|K}$ is a representation of K and $\pi \simeq n\delta$

(E, U) irreducible representation of G in a Banach space E

$$E = \oplus_{k \in \hat{K}} E(\delta), \quad P(\delta) = \int_K \chi_\delta(k^{-1}) U(k)\, dk$$

Theorem. If $\dim E(\delta) < \infty$ then

$$\Phi(g)a = P(\delta)U(g)a, \quad a \in E(\delta)$$

is a quasi bounded irreducible spherical function of type δ. The converse is also true.

Godement and Harish-Chandra (1952): $\phi(g) = tr(\Phi(g))$

T (1970, 1977), Gangolli and Varadarajan (1988)

Let G be a Lie group, (V, π) representation of K, $\pi \simeq n\delta$

Theorem

$\Phi : G \longrightarrow \mathrm{End}(V)$ continuous is spherical iff

i) Φ analytic, and $\Phi(e) = I$
ii) $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$
iii) $[D\Phi](g) = \Phi(g)[D\Phi](e)$, $D \in D(G)^K$

Moreover,

$$D \mapsto [D\Phi](e)$$

is a representation of $D(G)^K$ which caracterizes Φ up to equivalence

Matrix Hypergeometric Equation

$A, B, C \in \mathrm{End}(V)$.

$$z(1-z)F''(z) + (C - z(A+B+I))F'(z) - ABF(z) = 0.$$

Let

$${}_2F_1\left({A\,;\,B \atop C}\,;z\right) = \sum_{m=0}^{\infty} \frac{z^m}{m!}(C;A;B)_m,$$

where the symbol $(C;A;B)_m$ is defined inductively by

$$\begin{aligned} (C;A;B)_0 &= 1, \\ (C;A;B)_{m+1} &= (C+m)^{-1}(A+m)(B+m)(C;A;B)_m, \quad m \geq 0. \end{aligned}$$

Theorem (T, PNAS 2003) If no eigenvalue of C is an integer less or equal to zero, then ${}_2F_1\left({A\,;\,B \atop C} ; z \right)$ is analytic on $|z| < 1$, with values in $\mathrm{End}(V)$.

If $F_0 \in V$ then

$$F(z) = {}_2F_1\left({A\,;\,B \atop C} ; z \right) F_0$$

is the unique analytic solution at $z = 0$ of the hypergeometric equation such that $F(0) = F_0$.

More general hypergeometric equation. Given $U, V, C \in \mathrm{End}(V)$.

$$z(1-z)F''(z) + (C - zU)F'(z) - VF(z) = 0.$$

Let

$${}_2H_1\left({U\,;\,V \atop C} ; z \right) = \sum_{m=0}^{\infty} \frac{z^m}{m!} [C; U; V]_m,$$

where the symbol $[C;U;V]_m$ is defined inductively by

$$[C;U;V]_0 = 1,$$
$$[C;U;V]_{m+1} = [C+m]^{-1}(m^2 + m(U-1) + V)[C;U;V]_m, \quad m \geq 0.$$

Theorem (T, PNAS 2003) If no eigenvalue of C is an integer less or equal to zero, then ${}_2H_1\left({U\,;\,V \atop C}\,;z\right)$ is analytic on $|z|<1$, with values in $\mathrm{End}(V)$.

If $H_0 \in V$ then
$$H(z) = {}_2H_1\left({U\,;\,V \atop C}\,;z\right)H_0$$
is the unique analytic solution at $z=0$ of the hypergeometric equation such that $H(0) = H_0$.

Spherical functions associated to $P_2(C)$

To express, as in the scalar case, all irreducible spherical functions, of a given type, associated to $P_2(C) = SU(3)/U(2)$ in terms of the matrix hypergeometric function. Then, to build out of all these spherical functions a sequence of matrix orthogonal polynomials.

$$G = \mathrm{SU}(3), \quad K = \mathrm{U}(2), \qquad \Phi : G \longrightarrow \mathrm{End}(V_\pi)$$

$$\Phi(x)\Phi(y) = \int_K \chi_\pi(k^{-1})\Phi(xky)\, dk, \qquad \Phi(e) = I$$

Grünbaum, Pacharoni and T (JFA 2002); Román and T (IJM 2006)

First examples of matrix orthogonal polynomials which are eigenfunctions of a second order differential operator.

$$D(G)^K = D(G)^G \otimes D(K)^K.$$

In particular $D(G)^K$ is abelian, hence irreducible spherical functions are of height one.

1. Φ analytic,
2. $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$,
3. $[D\Phi](g) = \lambda_D \Phi(g)$, $D \in D(G)^G$.

$$D(G)^G = \mathbb{C}[\Delta_2, \Delta_3] \quad (\Delta_2 \text{ is the Casimir operator of } G)$$

π extends to a unique holomorphic representation of $GL(2, C)$. Let $A(g)$ be the 2×2 upper left block of g, and let

$$\mathcal{A} = \{g \in G : \det A(g) \neq 0\}.$$

Let $\Phi_\pi(g) = \pi(A(g)),\ g \in \mathcal{A}$. To Φ we associate the function

$$H(g) = \Phi(g)\,\Phi_\pi(g)^{-1}, \quad g \in G.$$

1. $H(e) = I$
2. $H(gk) = H(g)$
3. $H(kg) = \pi(k)H(g)\pi(k^{-1})$
4. H is an eigenfunction of certain matrix differential operators

H is a function on the affine space $\mathbb{C}^2 \subset P_2(\mathbb{C})$.

H is determined by its restriction to a cross section of the K-orbits in $\mathbb{C}^2$, which are the spheres of radios $r \in [0, \infty)$; H is a diagonal matrix or a vector valued function.

$$\pi(A) = \pi_{n,\ell}(A) = (\det A)^n A^\ell, \quad n \in Z, \quad \ell \in Z_{\geq 0}.$$

The irreducible spherical functions are orthogonal with respect to

$$\langle \Phi, \Psi \rangle = \int_G \mathrm{tr}(\Phi(g)\Psi(g)^*)\, dg$$

Δ_2 is a symmetric operator: $\langle \Delta_2 \Phi, \Psi \rangle = \langle \Phi, \Delta_2 \Psi \rangle$.

Change of variables $u = \frac{r^2}{1+r^2}$, $u \in [0, 1)$

$$\bar{D}H = u(1-u)H'' + (2 - uA_1)H' + \frac{1}{u}(B_0 - B_1 + uB_1)H,$$

$$\bar{E}H = u(1-u)MH'' + (C_1 - C_0 - uC_1)H' + \frac{1}{u}(D_0 + D_1 - uD_1)H.$$

$\bar{D}$ is symmetric with respect to

$$\langle H, K\rangle = \int_0^1 K^*(u)W(u)H(u)\,du$$

$$W(u) = \sum_0^{\ell} u(1-u)^{n+\ell-i}E_{i,i}.$$

Theorem The irreducible spherical functions Φ of $SU(3)$ of type (n,ℓ) correspond precisely to the simultaneous $C^{\ell+1}$-valued polynomial eigenfunctions H of the differential operators $\bar{D}$ and $\bar{E}$, such that $h_i(u) = (1-u)^{i-n-\ell}g_i(u)$ for all $n+\ell+1 \le i \le \ell$ with g_i polynomial and $H(0) = (1,\ldots,1)^t$.

To put this into the framework of orthogonal polynomials we conjugate D: look for a function $\Psi(u)$ such that $D = \Psi(u)^{-1}\bar{D}\Psi(u)$ be a *hypergeometric operator*

$$D = u(1-u)\frac{d^2}{du^2} + (C - uU)\frac{d}{du} - V$$

The function $\psi(u) = XT(u)$, where X is the Pascal matrix $X_{i,j} = \binom{i}{j}$ and $T(u)$ is the diagonal matrix $T(u)_{i,i} = u^i$, is such a function.

Theorem The irreducible spherical functions Φ of $SU(3)$ of type (n, ℓ) are in a one to one correspondence with the functions $F(u) = \psi(u)^{-1}H(u)$. These are precisely the simultaneous $C^{\ell+1}$-valued polynomial eigenfunctions of the differential operators D and $E = \psi^{-1}\bar{E}\psi$, such that $F(0) = (1, x_1, \ldots, x_\ell)^t$.

Therefore, if Φ is an irreducible spherical function on G of type (n, ℓ), then there exist $\lambda \in C$ and $F_0 = (1, x_1, \dots, x_\ell)^t$ such that

$$F(u) = {}_2H_1\left({U \, ; \, V+\lambda \atop C} ; u \right) F_0.$$

Hence, there exits a nonnegative integer w such that $[C; U; V]_{w+1}$ is singular and $[C; U; V]_w$ is not singular. This implies that

$$\lambda = -w(w + n + \ell + k + 2) - k(n + k + 1), \quad 0 \le k \le \ell.$$

Let

$$W_\lambda = \{F = C^{\ell+1}[u] : DF = \lambda F\}$$

Since $F(0)$ determines $F \in W_\lambda$, the linear map

$$\nu : W_\lambda \longrightarrow \mathbb{C}^{\ell+1}, \quad \nu(F) = F(0)$$

is a surjective isomorphism.

The differential operators D and E commute, since they come, respectively, from Δ_2 and Δ_3 of $D(G)^G$. Therefore E restricts to a linear operator of W_λ into itself. Then the following is a commutative diagram

$$\begin{array}{ccc} W_\lambda & \xrightarrow{E} & W_\lambda \\ \nu \big\downarrow & & \big\downarrow \nu \\ \mathbb{C}^{\ell+1} & \xrightarrow{M(\lambda)} & \mathbb{C}^{\ell+1} \end{array}$$

$M(\lambda)$ is the $(\ell+1)\times(\ell+1)$ matrix given by

$$M(\lambda) = Q_0(C+1)^{-1}(U+V+\lambda)C^{-1}(V+\lambda) + P_0C^{-1}(V+\lambda) + R.$$

Theorem For any (n, ℓ) and any $\lambda \in C$, the eigenvalues of $M(\lambda)$ are

$$\mu_j(\lambda) = \lambda(n - \ell + 3j) - 3j(\ell - j + 1)(n + j + 1),$$

$0 \le j \le \ell$, and all have geometric multiplicity one. Moreover, if $(v_0, \dots, v_\ell)^t$ is a nonzero eigenvector of $M(\lambda)$, then $v_0 \neq 0$.

Theorem If $\lambda = \lambda_{w,k} = -w(w+n+\ell+k+2)-k(n+k+1), \quad 0 \le k \le \ell$, there exits a unique μ-eigenvector $F_0 = (1, x_1, \dots, x_\ell)$ of $M(\lambda)$ such that

$$F_{w,k}(u) = {}_2H_1\left({U \,;\, V+\lambda \atop C} ; u \right) F_0.$$

is a polynomial of degree w. Moreover $\mu = \mu_k(\lambda)$.

Let

$$P_w = (F_{w,0}, \dots, F_{w,\ell}), \qquad w \ge 0.$$

Theorem $\{P_w\}$ is a sequence of orthogonal polynomials with respect to the weight function $\psi^* W \psi$, and $DP_w = P_w \Lambda(w)$, where Λ_w is the diagonal matrix with $\Lambda(w)_{k,k} = \lambda_{w,k}$.

The irreducible representations of $SU(3)$ are labeled by $\mathbf{m} = (m_1, m_2, m_3)$ with $m_1 \geq m_2 \geq m_3$, integers. Let V be the standard representation. Assume that $m_1 \geq n + \ell \geq m_2 \geq n \geq m_3$. Then by considering

$$V_{\mathbf{m}} \otimes V = V_{\mathbf{m}_1} \oplus V_{\mathbf{m}_2} \oplus V_{\mathbf{m}_3},$$

$\mathbf{m}_1 = (m_1 + 1, m_2, m_3)$, $\mathbf{m}_2 = (m_1, m_2 + 1, m_3)$, $\mathbf{m}_3 = (m_1, m_2, m_3 + 1)$,

we obtain the three term recursion relation for the sequence $\{P_w\}$:

$$u \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} B_0 & C_0 & 0 & \cdot & \cdot & \cdot \\ A_1 & B_1 & C_1 & 0 & \cdot & \cdot \\ 0 & A_2 & B_2 & C_2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}.$$

Project

To express, as in the scalar case, all irreducible spherical functions of a given type associated to a compact two point homogeneous space in terms of a sequence of matrix orthogonal polynomials given by the matrix hypergeometric function.

First step: The complex projective space

$$G = \mathrm{SU}(k+1), \quad K = \mathrm{U}(k), \quad G/K = P^k(\mathbb{C})$$

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