

Deterministic and Stochastic, Time and Space Signal Models: An Algebraic Approach

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Structure and Digital Signal Processing

■ Is DSP algebraic?

- By restricting to Linear Algebra are we missing something?
- Apparently disparate concepts \implies instantiations same concept

■ Is DSP geometric?

- Constraints may restrict signals to a manifold
- Algorithms and signal processing should be derived for manifolds

■ Special Session at IEEE International Conference on Signal Processing (ICASSP'06)

- DSP: Algebra vs. Geometry

References and URL's

- Markus Pueschel and José M. F. Moura, "*The Algebraic Approach to the Discrete Cosine and Sine Transforms and their Fast Algorithms*," **SIAM Journal of Computing**, vol 35:(5), pp. 1280-1316, March 2003.
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- <http://www.ece.cmu.edu/~smart>
- <http://www.spiral.net>

Algebraic Theory of SP

- Quick refresh on DSP
- DSP: Algebraic view point
 - Signal Model
- Algebraic Theory: Time
 - Time shift
 - Boundary conditions (finite time)
 - Fourier transforms, spectrum
- Algebraic Theory: Space
 - Space shift
 - Infinite space: C-transform and DSFT
 - Finite space: DTTs
- What is it useful for:
 - Fast algorithms
 - m-D: separable and non-separable, new transforms

DSP

- Scalar, discrete index (time or space) linear signal processing
- 1-D or m-D: indexing set
- Example: infinite discrete time

- Signals:

$$S(z) = \sum_{n \in \mathbb{Z}} s_n z^{-n} \in \left\{ \sum_{n \in \mathbb{Z}} s_n z^{-n} \mid (\dots, s_{-1}, s_0, s_1, \dots) \in \ell^2(\mathbb{Z}) \right\} = \mathcal{M}$$

- Filters:

$$H(z) = \sum_{n \in \mathbb{Z}} h_n z^{-n} \in \left\{ \sum_{n \in \mathbb{Z}} h_n z^{-n} \mid (\dots, h_{-1}, h_0, h_1, \dots) \in \ell^1(\mathbb{Z}) \right\} = \mathcal{A}$$

- Convolution (multiplication):

$$Y(z) = H(z) \cdot S(z) = \sum_{n \in \mathbb{Z}} \left(\sum_{i \in \mathbb{Z}} h_i s_{n-i} \right) z^{-n} \in \mathcal{M}$$

- z-Transform:

$$\Phi : \quad V = \ell^2(\mathbb{Z}) \quad \rightarrow \quad \mathcal{M}$$

$$s = (\dots, s_{-1}, s_0, s_1, \dots) \quad \mapsto \quad \sum_{n \in \mathbb{Z}} s_n z^{-n}$$

DSP

- Fourier Transform: DTFT


$$S(\omega) = S(e^{j\omega}) = \sum_{n \in \mathbb{Z}} s_n e^{-j\omega n}; \quad s_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) e^{j\omega n} d\omega$$

- Spectrum: $(S(\omega))_{\omega \in W = [-\pi, \pi)}$

- Impulses: $E_\omega(z) = \sum_{n \in \mathbb{Z}} e^{j\omega n} z^{-n}$

- Eigen property: $H(z)E_\omega(z) = H(e^{j\omega n})E_\omega(z)$

- Linear combination: $\alpha H(z) + \beta H'(z); \quad \alpha S(z) + \beta S'(z)$

-  \mathcal{A} and \mathcal{M} are **vector spaces**

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DSP: Algebraic View Point

- Cascading of filters: $H(z) \cdot H'(z)$
- \Longrightarrow makes \mathcal{A} an algebra – the algebra of filters
- Convolution (multiplication): $Y(z) = H(z) \cdot S(z)$
- \Longrightarrow makes \mathcal{M} an \mathcal{A} -module – the module of signals
- Signal Model: Triplet $(\mathcal{A}, \mathcal{M}, \Phi)$
 - where bijective linear mapping $\Phi : V \rightarrow \mathcal{M}$

DSP: Finite Time

- Signals: $S(z) = \sum_{i=0}^{n-1} s_i z^{-i} \in \mathcal{M} \stackrel{?}{=} \{\sum_{i=0}^{n-1} s_i z^{-i}\}$
- Filters: $H(z) = \sum_{i=0}^{n-1} h_i z^{-i} \in \mathcal{A} \stackrel{?}{=} \{\sum_{i=0}^{n-1} h_i z^{-i}\}$
- Convolution (multiplication): $Y(z) = H(z) \cdot S(z) \notin \mathcal{M}$
- Candidates: algebras of filters \mathcal{A} and modules of signals \mathcal{M} ?

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Algebraic Theory: Shift

- Shift: special type of filter $x \in \mathcal{A}$
 - $h \in \mathcal{A}$ expressed as polynomial or series in x
 - $\implies x$ generates \mathcal{A}
- Shift invariance: x shift, $\forall s, h : h(xs) = x(hs) \implies hx = xh$
 - Since x is shift, \mathcal{A} is commutative, so this is trivially verified
 - Conversely, \mathcal{A} comm., x generates \mathcal{A} , then all filters are shift-inv.
- Which algebras are shift invariant (comm. & generated by single x ?)
 - Infinite case: series in x or polynomials in x
 - Finite dimensional case: *polynomial algebras*, $p(x)$ polyn. deg n
 $\mathcal{A} = \mathbb{C}[x]/p(x) = \{\text{polyn. deg} < n, \text{ addition, mult. mod } p(x)\}$

- Signal Model: finite dimensional case

$$(\mathcal{A} = \mathbb{C}[x]/p(x), \mathcal{M} = \mathbb{C}[x]/p(x), \Phi)$$

$$\Phi : V = \mathbb{C}^n \rightarrow \mathcal{M}$$

Algebraic Theory: Infinite Time

■ Realization of signal model (infinite time):

- Time marks and shift operator (Kalman 68): $q \diamond t_n = t_{n+1}$
- k -fold shift: $q_k \diamond t_n = t_{n+k}, \implies q_k = q^k$
- Linear extension:
 - Extend q from t_n to set $\{\sum s_n t_n\}$
 - Extend from q^k to set of all formal sums $\{\sum h_k q^k\}$
 - Realization: set $q = x$ and $\diamond = \cdot \implies t_{n+1} = x \cdot t_n$
 - Two-term recursion solution: $t_0 = 1, t_n = x^n$

■ $\implies \mathcal{M} = \{s = \sum s_n x^n\}$ and $\mathcal{A} = \{h = \sum h_k x^k\}$

$$\Phi : s \mapsto \sum_{n \in \mathbb{Z}} x^n$$

Remark: we use x rather than z^{-1}

Algebraic Theory: Finite Time

■ Realization of signal model (finite time):

- Problem: $x \cdot x^{n-1} = x^n \notin \mathbb{C}_n[x] = \left\{ \sum_{i=0}^{n-1} s_i x^i \right\}$
 $x^{-1} \cdot x^0 = x^{-1} \notin \mathbb{C}_n[x] = \left\{ \sum_{i=0}^{n-1} s_i x^i \right\}$

■ Boundary condition and signal extension:

- $x^n = r(x) = \sum_{0 \leq k < n} \beta_k x^k$, or $x^n - r(x) = 0$

Equivalent to right b.c. $s_n = \sum_{0 \leq k < n} \beta_k s_k$

Replaces vector space $\mathbb{C}_n[x]$ by $\mathcal{M} = \mathbb{C}[x] / (x^n - r(x))$

$$x^{-1} \in \mathcal{A} \text{ iff } x \nmid r(x) \implies x^{-1} = -\frac{1}{\beta_0} (\beta_1 + \beta_2 x + \cdots + \beta_{n-2} x^{n-2} - x^{n-1})$$

b.c. \implies Right and left signal extension

- Signal model:

$$(\mathcal{A} = \mathcal{M} = \mathbb{C}[x] / (x^n - r(x)), \Phi(\mathbf{s}) = \sum_{0 \leq k < n} s_k x^k \in \mathcal{M})$$

Finite Time and DFT

■ Signal model: $(\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/(x^n - 1), \sum_{0 \leq k < n} s_k x^k \in \mathcal{M})$

■ Fourier transform: DFT

$$\Delta : \mathcal{M} = \mathbb{C}[x]/(x^n - 1) \rightarrow \bigoplus_{0 \leq k < n} \mathbb{C}[x]/(x - \omega_n^k)$$

$$\omega_n = e^{-2\pi j/n}$$

■ In matrix format: $\text{DFT}_n = [\omega_n^{kl}]_{0 \leq k, l < n}$

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Space Signal Model: Space Shift

■ Shift: symmetric definition $q \diamond t_n = (t_{n+1} + t_{n-1}) / 2$

k -fold shift: $q_k \diamond t_n = (t_{n+k} + t_{n-k}) / 2, \implies q_k = q^k$

Differences wrt time model: $q_k \neq q^k, q_k = q_{-k}$

Lemma: The k -fold space shift operator is $q_k = T_k(q)$, the Chebyshev polynomials of the 1st kind

Linear extension: extend operation of q to

$$\mathcal{A} = \left\{ \sum_{k \geq 0} h_k q_k = \sum h_k T_k(q) \right\}, \quad \mathcal{M} = \left\{ \sum s_n t_n \right\}$$

Realization: $q = x$ and $\diamond = \cdot$

time marks t_n , i.e., satisfy $x \cdot C_n = (C_{n+1} + C_{n-1}) / 2$

$\implies C_n$ sequence of Chebyshev polyn., $\{T_n, U_n, V_n, W_n\}$

Signal Model: Infinite Space

- Signal Model:

$$\left(\mathcal{A} = \left\{ h = \sum_{k \geq 0} h_k T_k \right\}, \mathcal{M} = \left\{ s = \sum_{n \geq 0} s_n C_n \right\}, \Phi \right)$$

- C-transform: $\Phi : V = \ell^2(\mathbb{N}) \rightarrow \mathcal{M} = \left\{ \sum_{n \geq 0} s_n C_n \right\}$

$$\Phi(s) = \sum_{n \geq 0} s_n C_n, \quad C_n \in \{T_n, U_n, V_n, W_n\}$$

- Follows from property of Chebyshev polyn.: k -fold shift

$$T_k \cdot C_n = (C_{n+k} + C_{n-k}) / 2$$

- Fourier transform: DSFT, e.g., choose $C_n = T_n$

$$\Delta : \quad \mathcal{M} \rightarrow \left(\bigoplus_{\omega \in [0, \pi]} \mathbb{C} \right) = \mathbb{C}^{[0, \pi]}$$

$$s = S(x) \mapsto S(\cos \omega)_{\omega \in [0, \pi]} = \omega \mapsto S(\cos \omega).$$

Signal Model: Finite Space

- Left b.c.: $C_1 = ax + b, a \neq 0, \implies C_{-1} = \frac{2-a}{a}C_1 - \frac{2b}{a}C_0$

- Monomial signal extension: $C \in \{T, U, V, W\}$

- Right b.c.: problem with

$$T_1 \cdot C_{n-1} = x \cdot C_{n-1} = (C_{n-2} + C_n)/2 \notin \left\{ \sum_{0 \leq k < n} s_k C_k \right\}$$

- Lemma (Monomial right sig. extension): Let $C \in \{T, U, V, W\}$

Only 4 right bc yield monomial right sig. ext. for $\mathcal{M} = \mathbb{C}[x]/p(x)$

$$C_n = C_{n-2}, C_n = 0, C_n = \pm C_{n-1}$$

$$\implies p \in \{C_n - C_{n-2}, C_n, C_n \pm C_{n-1}\}$$

$$\implies 16 \text{ possibilities}$$

Finite Sp.Signal Model: Finite C-transf. & DTTs

- Let seq. Chebyshev poly.: C_0, \dots, C_{n-1} , $C \in \{T, U, V, W\}$
- Let: $p \in \{C_n - C_{n-2}, C_n, C_n \pm C_{n-1}\}$

	$C_n - C_{n-2}$	C_n	$C_n - C_{n-1}$	$C_n + C_{n-1}$
T_n	$2(x^2 - 1)U_{n-2}$	T_n	$(x - 1)W_{n-1}$	$(x + 1)V_{n-1}$
U_n	$2T_n$	U_n	V_n	W_n
V_n	$2(x - 1)W_{n-1}$	V_n	$2(x - 1)U_{n-1}$	$2T_n$
W_n	$2(x + 1)V_{n-1}$	W_n	$2T_n$	$2(x + 1)U_{n-1}$

- 16 finite space signal models: $(\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/p(x), \Phi)$

- Finite C-transform: $\Phi : V = \mathbb{C}^n \rightarrow \mathcal{M} = \mathbb{C}[x]/p(x)$

$$\Phi(\mathbf{s}) = \sum_{0 \leq k < n} s_k C_k$$

Finite Sp. Sig. Model: Finite C-transf. & DTTs

- Fourier transforms: 16 DTTs (8 DCTs and 8 DSTs)

- Example: Signal model for DCT, type 2

- Left bc: $s_{-1} = s_0 \implies C_1 = C_0 = 1$ afforded by $C = V$

- Right bc: $s_n = s_{n-1} \implies C_n = C_{n-1}$

$$\implies p = C_n - C_{n-1} = V_n - V_{n-1} = 2(x-1)U_{n-1}$$

- Sig. model for DCT, type 2:

$$(\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/(x-1)U_{n-1}, \Phi)$$

- DCT, type 2:

$$\Delta : \mathbb{C}[x]/(x-1)U_{n-1} \rightarrow \bigoplus_{0 \leq k < n} \mathbb{C}[x]/(x - \alpha_k)$$

$$\text{Zeros of } (x-1)U_{n-1}, \alpha_k = \cos k\pi/n, 0 \leq k < n$$

$$\text{DCT-}2_n = \text{diag}_{0 \leq k < n}(\cos k\pi/(2n)) \cdot \left[\frac{1}{\cos k\pi/(2n)} \cdot \cos k(\ell + 1/2)\pi/n \right]$$

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Fast Algorithms: DTTs

- DTTs: DCT, type 2: Direct sum: fast alg. Via poly. factorization

$\mathbb{C}[x]/(x-1)U_{n-1}$ with basis $b = (V_0, \dots, V_{n-1})$

Property of U : $U_{2m-1} = 2 \cdot U_{m-1} \cdot T_m$

$$\mathbb{C}[x]/(x-1)U_{2m-1} \cong \mathbb{C}[x]/(x-1)U_{m-1} \oplus \mathbb{C}[x]/T_m$$

$$\text{DCT-2}_{2m} = L_m^{2m} \cdot (\text{DCT-2}_m \oplus \text{DCT-4}_m) \cdot B.$$

$$B = \begin{bmatrix} 1 & & & & 1 \\ & \cdot & & & \\ & & 1 & 1 & \\ 1 & & & & -1 \\ & \cdot & & & \\ & & 1 & -1 & \end{bmatrix} = \begin{bmatrix} I_m & J_m \\ I_m & -J_m \end{bmatrix} = (\text{DFT}_2 \otimes I_m)(I_m \oplus J_m)$$

Fast Algorithms: DTTs

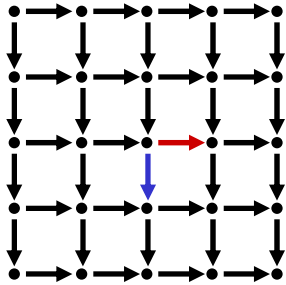
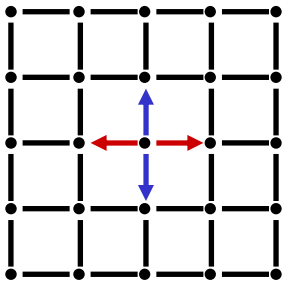
THEOREM 8.6. *The following recursive algorithms for DTTs are based on the rational factorization $U_{2m-1} = 2 \cdot U_{m-1} \cdot T_m$. We also indicate where they first appeared in the literature (to our best knowledge).*

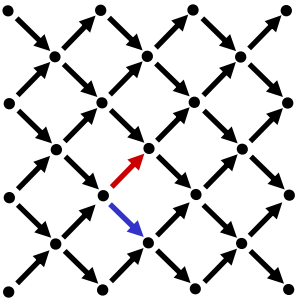
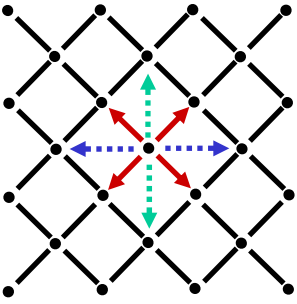
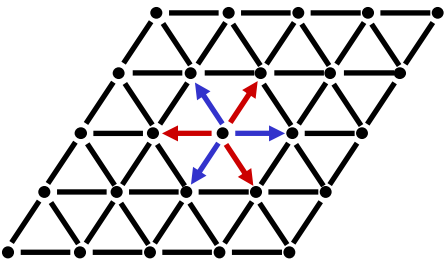
- (i) $\text{DCT-1}_{2m+1} = P_{2m+1} \cdot (\text{DCT-1}_{m+1} \oplus \text{DCT-3}_m) \cdot B_{2m+1}$, [27].
- (ii) $\text{DST-1}_{2m-1} = P_{2m-1} \cdot (\text{DST-3}_m \oplus \text{DST-1}_{m-1}) \cdot B_{2m-1}$, [55].
- (iii) $\text{DCT-2}_{2m} = P_{2m} \cdot (\text{DCT-2}_m \oplus \text{DCT-4}_m) \cdot B_{2m}$, [7].
- (iv) $\text{DST-2}_{2m} = P_{2m} \cdot (\text{DST-4}_m \oplus \text{DST-2}_m) \cdot B_{2m}$, [52].

THEOREM 8.7. *The following recursive algorithms for DTTs are based on the rational factorization $U_{2m} = V_m W_m$.*

- (i) $\text{DCT-1}_{2m} = P_{2m} \cdot (\text{DCT-5}_m \oplus \text{DCT-7}_m) \cdot B_{2m}$.
- (ii) $\text{DST-1}_{2m} = P_{2m} \cdot (\text{DST-7}_m \oplus \text{DST-5}_m) \cdot B_{2m}$.
- (iii) $\text{DCT-2}_{2m+1} = P_{2m+1} \cdot (\text{DCT-6}_{m+1} \oplus \text{DCT-8}_m) \cdot B_{2m+1}$.
- (iv) $\text{DST-2}_{2m+1} = P_{2m+1} \cdot (\text{DST-8}_{m+1} \oplus \text{DST-6}_m) \cdot B_{2m+1}$.

Finite Signal Models in Two Dimensions

Visualization (without b.c.)	Signal Model $\mathcal{A} = \mathcal{M}$	Fourier Transform
 <p>time, separable</p>	$\mathbb{C}[x, y] / \langle x^n - 1, y^n - 1 \rangle$ time shifts: x, y	$\text{DFT}_n \otimes \text{DFT}_n$
 <p>space, separable</p>	for example $\mathbb{C}[x, y] / \langle T_n(x), T_n(y) \rangle$ space shifts: x, y	$\text{DCT}_n \otimes \text{DCT}_n$ (16 types)

 <p>time, nonseparable</p>	$\mathbb{C}[u, v] / \langle u^n - 1, u^{\frac{n}{2}} - v^{\frac{n}{2}} \rangle$ <p>time shifts: u, v</p>	$\text{DDQT}_{n \times \frac{n}{2}}$ <p>Püschel ICASSP '05 (separable, Mersereau)</p>
 <p>space, nonseparable</p>	$\mathbb{C}[u, v, w] / \langle T_{n/2}(u), T_{n/2}(v), 4w^2 - (u+1)(v+1) \rangle$ <p>space shifts: u, v, w</p>	$\text{DQT}_{n \times \frac{n}{2}}$ <p>Püschel ICIP '05</p>
 <p>space, nonseparable</p>	$\mathbb{C}[x, y] / \langle C_n(x, y), \overline{C}_n(x, y) \rangle$ <p>space shifts: u, v</p>	$\text{DTT}_{n \times n}$ <p>Püschel ICASSP '04</p>

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