Orthogonal Polynomials in one and several variables and applications

Auto regressive models in 1D

$$s_n = -\sum_{k=1}^{q} a_k s_{n-k} + \sigma u_n,\ \sigma > 0 \qquad (1)$$

Random models

Weakly stationary (2nd order) processes $\{X_i\}_{i=-\infty}^{\infty}$.

$i\}$ $X_i \in L^2(P)$,

$ii\}$ $E(X_i) = m$ for all i,

$iii\}$ $c_h = E(X_{i+h}, \bar{X}_i)$ for all i.

White Noise process $Z(0, \sigma^2)$

$m = 0,\ c_h = \left\{ \begin{array}{ll} 0 & h \neq 0 \\ \sigma^2 & h = 0 \end{array} \right.$

Autoregressive process AR(q) Let $\{u_n\}$ be a $Z(0, 1)$ process. The process $\{s_n\}$ is an AR(q) process if it is a weakly stationary with zero mean and satisfies (1).

An AR(q) process is said to be *causal* if there exists a sequence $\{\psi_j\} \in l^1$ such that $s_n = \sum_{j=0}^{\infty} \psi_j u_{n-j} \ \forall n$

Note s_n is independent of $u_j, j > n$.

It follows from Fourier series that an AR(q) process has a causal solution if and only if

$$p(z) = 1 + \sum_{k=1}^{q} a_k z^k$$

is a stable polynomial.

Let $U = \{z : |z| < 1\}$. A polynomial $p(z)$ is said to be stable if $p(z) \neq 0, z \in \bar{U}$.

Typically you are given or you estimate a finite number of correlation (autocovariance) coefficients $\{c_h\}_0^q,\ c_{-h} = \bar{c}_h$.

What conditions must these correlation coefficients satisfy in order for there to be a causal AR(q) process $\{s_n\}$ with $c_h = E(s_{n+h}, \bar{s}_n)\ h = 0 \dots q$.

Arrive at Yule-Walker equations

$$\begin{bmatrix} c_0 & \bar{c}_1 & \cdots & \bar{c}_q \\ \vdots & \ddots & \ddots & \vdots \\ c_q & c_{q-1} & \cdots & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ a_q \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}.$$

Toeplitz because of stationarity.

$$C_q = \begin{bmatrix} c_0 & \bar{c}_1 & \cdots & \bar{c}_q \\ \vdots & \ddots & \ddots & \vdots \\ c_q & c_{q-1} & \cdots & c_0 \end{bmatrix}.$$

Since the above matrix is positive semidefinite there is a unique solution if and only if it is positive definite.

How to see stability?

Let μ be a positive Borel measure support on the unit circle with an infinite number of points of increase. Let $\{\phi_i(z)\}, i = 0, \ldots, n$, be the unique sequence of polynomials such that $\phi_i(z)$ is a polynomial of degree i in z with positive leading coefficient and $\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_k(e^{i\theta})\overline{\phi_j(e^{i\theta})}d\mu(\theta) = \delta_{k,j}$.

Orthogonality implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overleftarrow{\phi}_n(e^{i\theta})z^{-j}d\mu = 0,\ 0 < j \leq n\ z = e^{i\theta}$$

Recurrence formula

The polynomials ϕ_i above satisfy the following difference equations.

$$a(n)\phi_n(z) = z\phi_{n-1}(z) - \alpha_n \overleftarrow{\phi}_{n-1}(z),$$

and

$$a(n)\overleftarrow{\phi}_n(z) = \overleftarrow{\phi}_{n-1}(z) - z\bar{\alpha}_n\phi_{n-1}(z),$$

where $\overleftarrow{\phi}_n(z) = z^n\bar{\phi}(1/z)$,

$$a(n) = \frac{k_{n-1}}{k_n}.$$

The coefficients $\{\alpha_n\}$ are called recurrence coefficients. Note

$$a(n)^2 = 1 - |\alpha_n|^2,$$

so that

$$|\alpha_n| < 1.$$

From recurrence formula we find

$$\overleftarrow{\phi_n}(z)\overline{\overleftarrow{\phi_n}(z_1)} - \phi_n(z)\overline{\phi_n(z_1)}$$
$$= \overleftarrow{\phi}_{n-1}(z)\overline{\overleftarrow{\phi_{n-1}}(z_1)} - z\bar{z}_1\phi_{n-1}(z)\overline{\phi_{n-1}(z_1)}$$

Christoffel-Darboux formula,

$$\frac{\overleftarrow{\phi_n}(z)\overline{\overleftarrow{\phi_n}(z_1)} - \phi_n(z)\overline{\phi_n(z_1)}}{(1 - z\bar{z}_1)} = \sum_{i=0}^{n-1} \phi_i(z)\overline{\phi_i(z_1)} = K_n(z, z_1)$$

and

$$\frac{\overleftarrow{\phi_n}(z)\overline{\overleftarrow{\phi_n}(z_1)} - z\bar{z}_1\phi_n(z)\overline{\phi_n(z_1)}}{(1 - z\bar{z}_1)} = \sum_{i=0}^{n} \phi_i(z)\overline{\phi_i(z_1)}.$$

Also

$$\sum_{i=0}^{n} \phi_i(z)\overline{\phi_i(z_1)} = [1, z, z^2..z^n]C_n^{-1}[1, z_1...z_1^n]^*.$$

Follows from Cholesky factorization.

$$C_n^{-1} = U_n U_n^*$$

multiplication by $[1, z...z^n]$ and $[1, z_1...z_1^n]^*$ gives result.

Properties

1) $\overleftarrow{\phi_n}$ is stable (no zeros inside and on the unit circle)

2) $c_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ij\theta} d\mu = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{-ij\theta}}{|\overleftarrow{\phi}_n|^2} d\theta, |j| \leq n$

Relation to orthogonal polynomials.

From properties assumed on s_n imply

$$\mathcal{E}(s_n \overline{s_{n-j}}) = c_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ij\theta} d\rho(\theta)$$

ρ a positive measure supported on the unit circle. Let $\phi_0, \phi_1 \cdots$ be its orthonormal polynomials. Then

$$p(z) = l_q \overleftarrow{\phi_q}(z).$$

If $Q_n(\theta)$ is a strictly positive trignometric polynomial of degree n Then $d\mu = \frac{1}{2\pi} \frac{1}{Q} d\theta$ is called a Bernstein-Szego measure and from above is given by $d\mu = \frac{1}{2\pi} \frac{1}{|\overleftarrow{\phi_n(z)}|^2} d\theta$

Definition

$$H^\infty = \{f \in H(U), \sup|f(z)| < \infty,\ z \in U\}$$

$$||f||_\infty = \sup|f(z)| < \infty,\ z \in U\}$$

f is an inner function if $f \in H^\infty$ and $\lim_{r\to 1}|f(re^{i\theta}| = 1$ for almost everywhere θ.

The Christoffel-Darboux gives

$$1 - f(z)\overline{f(w)} = \sum_{k=0}^{n-1} f_i(z)(1 - z\bar{w})\overline{f_i(w)}.\ z, w \in U$$

where f is a rational inner function of degree n and f_i are rational function with the same denominator as f.

This gives a simple proof of Von Neumann's inequality.

Let H be a Hilbert space. An operator T acting on H is said to be a contraction if $||T|| \leq 1$. T is said to be a strict contraction if $||T|| < 1$.

Theorem Let T be a strict contraction on a Hilbert space H. If P is a polynomial then

$$||P(T)|| \leq \sup_U |P|.$$

Proof(Cole-Wermer). T is a strict contraction if and only if $I - TT^* > 0$. Let P be a polynomial in z with $|P(z)| \leq 1$ for $|z| \leq 1$. By Caratheorory's theorem there is a sequence of rational inner functions $\{f_m\}$ such that $f_m \to P$ uniformly on each disk $|z| \leq r < 1$. Since T is a strict contraction there is and $r < 1$ such that $||T|| < r$. From the Christoffel-Darboux formula

$$I - f_m(T)f_m(T)^* = \sum_{k=0}^{n-1} \tilde{f}_{i,m}(T)(1 - TT^*))\tilde{f}_{i,m}(T)^* \geq 0,$$

so that $||f_m(T)|| \leq 1$ which gives the result.

What about 2D?

Many things not true.

$AR(\bar{p})$ model.

$$s_{n,m} = -\sum_{k=0}^{p}\sum_{l=0}^{q} a_{k,l} s_{n-k,m-l} + \sigma u_{n,m},$$

$(k,l) \neq (0,0), u_{n,m}$ is a doubly indexed sequence of mean zero, unit variance white noise random variables.

In order to talk about causal solutions need a notion of forward and past.

Helson- Lowdenslager half plane S

1. $(0,0) \notin S$
2. if $(n,m) \in S$ then $(-n,-m) \notin S \quad (n,m) \neq (0,0)$
3. if (n,m) and (n_1,m_1) in S then $(n+n_1, m+m_1) \in S$

$$S = \{(n,m), \{-\infty < m < \infty, n > 0\} \cup \{0 < m, n = 0\}\}$$

$AR(\bar{p})$ is a causal process if

$$s_{n,m} = \sum_{(k,l) \in S \cup \{0,0\}} \beta_{k,l} u_{n-k,m-l}.$$

where

$\sum |\beta_{k,l}| < \infty$

Let

$$U^n = \{z = (z_1, ...z_n), z_i \in C \ |z_i| < 1, \ 1 \leq i \leq n\},$$

$$T^n = \{z = (z_1, ...z_n), z_i \in C \ |z_i| = 1, \ 1 \leq i \leq n \ \}$$

For $z \in C^n$ and $k \in Z^n$, $z^k = z_1^{k_1} z_2^{k_2} ... z_n^{k_n}$.

A polynomial $p(z), z \in C^n$ is said to be stable if

$p(z) \neq 0 \ |z_i| \leq 1, \ 0 \leq i \leq n$

As above a causal solution exists if and only if

$$p(z, w) = 1 + \sum_{k=0}^{p} \sum_{l=0}^{q} a_{k,l} z^k w^l$$

$(k, l) \neq (0, 0)$ is stable.

Proposition Let $p(z,w)$ be a polynomial of degree (n,m). The following are equivalent

a. p is stable,

b. $\widehat{p^{-1}}(k,\ell) = 0$ for all $(k,\ell) \in \{(k,\ell) : k < 0 \text{ or } (k = 0 \text{ and } \ell < 0)\}$,

c. $\widehat{p^{-1}}(k,\ell) = 0$ for all $(k,\ell) \in \{(k,\ell) : k < 0 \text{ or } \ell < 0\}$.

Here the (k,ℓ) Fourier coefficient of $f(z,w)$ is denoted as $\widehat{f}(k,\ell)$.

We show that b$\Rightarrow$ c.

Let $p(z,w) = \sum_{j=0} p_j(w)z^j$ and $p^{-1} = \sum_{j\geq 0} q_j(w)z^j$. Then $\widehat{q_0}(\ell) = 0$ for $\ell < 0$. Since $pp^{-1} = 1$, we find

$$p_0(w)q_0(w) = 1$$

and

$$\sum_{\ell=0}^{j} p_{j-\ell}(w)q_\ell(w) = 0$$

By induction suppose $q_k(\ell) = 0,\ \ell < 0$ then

$$q_{k+1}(w) = -\frac{1}{p_0(w)} \sum_{\ell=0}^{k} p_{k+1-\ell}(w) q_\ell(w),$$

$$= -q_0(w) \sum_{\ell=0}^{k} p_{k+1-\ell}(w) q_\ell(w).$$

The correlation coefficients are

$$c_{k,l} = \mathcal{E}(s_{n+k,m+l}\overline{s_{n,m}}).$$

and

$$c_{-k,-l} = \bar{c}_{k,l}.$$

Conditions on correlation coefficient matrix in order to obtain a causal solution are more complicated.

Theory of bivariate orthogonal polynomials will be useful.

If

$$\mathcal{E}(|\sum_{i,j} s_{i,j} z^i w^j|^2) > 0,$$

for all finite sums Bochner's Theorem says that there is a two variable Borel measure μ on the bicircle such such that

$$c_{k,l} = \int_{T^2} e^{-ik\theta} e^{-il\phi} d\mu(\theta, \phi).$$

Useful to introduce two orderings

The *lexicographical ordering* which is defined by

$$(k,l) <_{lex} (k_1,l_1) \iff k < k_1 \text{ or } (k = k_1 \text{ and } l < l_1).$$

We shall also use the *reverse lexicographical ordering* which is defined by

$$(k,l) <_{revlex} (k_1,l_1) \iff (l,k) <_{lex} (l_1,k_1).$$

If we form the $(n+1)(m+1) \times (n+1)(m+1)$ moment matrix $C_{n,m}$ in the lexicographical ordering then it has the special form

$$C_{n,m} = \begin{pmatrix} C_0 & C_{-1} & \cdots & C_{-n} \\ C_1 & C_0 & \cdots & C_{-n+1} \\ \vdots & & \ddots & \vdots \\ C_n & C_{n-1} & \cdots & C_0 \end{pmatrix}$$

where each C_i is a $(m+1)\times(m+1)$ matrix of the form

$$C_i = \begin{pmatrix} c_{i,0} & c_{i,-1} & \cdots & c_{i,-m} \\ \vdots & & \ddots & \vdots \\ c_{i,m} & & \cdots & c_{i,0} \end{pmatrix}, \qquad i = 0,\ldots,n,$$

and

$$C_{-n} = C_n^*.$$

Thus $C_{n,m}$ is a block Toeplitz matrix where each block is a Toeplitz matrix so it has a doubly Toeplitz structure.

Appeal to theory of two variable orthogonal polynomials and Matrix orthogonal polynomials on the unit circle. Follow Delsarte et.al.

We perform the Gram-Schmidt procedure on the monomials

$$\{z^i w^j,\ 0 \le i \le n,\ 0 \le j \le m\}$$

using the lexicographical ordering and define the orthonormal polynomials $\phi^l_{nn,m}(z,w),\ 0 \le nn \le n, 0 \le l \le m$, by the equations ($z = e^{i\theta}, w = e^{i\phi}$),

$$\int_{T^2} \phi^l_{nn,m} z^{-i} w^{-j} d\mu(\theta,\phi) = 0,$$

for $0 \le i < nn, 0 \le j \le m$ or $i = nn, 0 \le j < l$ with

$$\int_{T^2} \phi^l_{nn,m} \overline{\phi^l_{nn,m}} d\mu(\theta,\phi) = 1,$$

and

$$\phi^l_{nn,m}(z,w) = k^{nn,l}_{nn,m,l} z^{nn} w^l + \sum_{(i,j) <_{\text{lex}} (nn,l)} k^{i,j}_{nn,m,l} z^i w^j.$$

With the convention $k^{nn,l}_{nn,m,l} > 0$, above equations uniquely specify $\phi^{l}_{nn,m}$.

Polynomials orthonormal obtained using the reverse lexicographical ordering will be denoted by $\tilde{\phi}^{l}_{n,mm}$

Set,

$$\Phi_{i,m} = \begin{pmatrix} \phi_{i,m}^{m} \\ \phi_{i,m}^{m-1} \\ \vdots \\ \phi_{i,m}^{0} \end{pmatrix}$$

Note that

$$\int_{T^2} \Phi_{i,m}\ z^{-k} w^{-j} d\mu = 0\ 0 \le k < i,\ 0 \le j \le m$$

If we write

$$\Phi_{i,m} = L_i^m(z)[w^m, w^{m-1}, \ldots, 1]^T,$$

then $L_i^m(z),\ i = 0, 1..n$ satisfy

$$\int_T L_i^m(z) dM_m(\theta) L_j^m(z)^* = I_{m+1}\delta_{i,j}\ z = e^{i\theta}$$

Then $L_i^m\ i = 0..n$ are left matrix polynomials orthonormal with respect to the measure

$$dM_m(\theta) = \int_T [1, w..w^m]^* d\mu_\theta(\phi)[1, w..w^m],$$

where $d\mu_\theta(\phi) = d\mu(\theta, \phi)$ and $w = e^{i\phi}$ so that dM_m is a Toeplitz matrix.

Matrix orthogonal polynomials

In general the are left matrix orthogonal polynomials as above but also right matrix orthogonal polynomials $R_i^m(z)$ which satisfy

$$\int_T R_i^m(z)^* dM_m(\theta) R_j^m(z) = I_{m+1}\delta_{i,j} \; z = e^{i\theta}$$

Also $\overleftarrow{L}_j^m(z) = z^j L_j^m(1/\bar{z})^*$ is stable. i.e. $det(\overleftarrow{L}_j^m(z)) \neq 0 |z| \leq 1$. The same is true of $\overleftarrow{R}_j^m(z)$. This follows from the matrix Christoffel-Darboux formula.

In the case above $L_i^m(z)^T = J R_i^m(z) J$ where J is an $(m+1) \times (m+1)$ matrix with ones down the reverse diagonal and zeros everywhere else.

What does doubly Toeplitz structure give

Recurrence relations

$$A_{n,m}\Phi_{n,m} = z\Phi_{n-1,m} - \hat{E}_{n,m}\overleftarrow{\Phi}^{T}_{n-1,m}$$

$$\Gamma_{n,m}\Phi_{n,m} = \Phi_{n,m-1} - K_{n,m}\tilde{\Phi}_{n-1,m},$$

$$\Gamma^1_{n,m}\Phi_{n,m} = w\Phi_{n,m-1} - K^1_{n,m}\overleftarrow{\tilde{\Phi}}^{*}_{n-1,m},$$

$$\Phi_{n,m} = I_{n,m}\tilde{\Phi}_{n,m} + \Gamma^*_{n,m}\Phi_{n,m-1},$$

$$\overleftarrow{\Phi}^{T}_{n,m} = I^1_{n,m}\tilde{\Phi}_{n,m} + (\Gamma^1_{n,m})^T\overleftarrow{\Phi}^{T}_{n,m-1}$$

Orthogonality gives various relations among matrices. For instance

$$\Gamma_{n,m}\Gamma^*_{n,m} = I - K_{n,m}K^*_{n,m}$$

Matrix Christoffel-Darboux formula

$$H_{n,m} = (1 - z\bar{z}_1)\tilde{\Phi}_{n,m}(z_1, w_1)^T\tilde{\Phi}^*_{n,m}(z, w)^T + H_{n,m-1}$$

where

$$H_{n,m}$$

$$= \overleftarrow{\Phi}_{n,m}(z_1, w_1)\overleftarrow{\Phi}^{*}_{n,m}(z, w) - z\bar{z}_1\Phi^T_{n,m}(z_1, w_1)\Phi^*_{n,m}(z, w)^T$$

$$H_{n,m} = [1, w...z^n w^m] C_{n,m}^{-1} [1, w_1..z_1^n w_1^m]^*$$

$$= [1, z...z^n w^m] \tilde{C}_{n,m}^{-1} [1, z_1...z_1^n w_1^m]^*$$

What about stability. If

$$0 = K_{n,m} = \int_{T^2} \tilde{\Phi}_{n-1,m} \Phi_{n,m-1}^* d\mu$$

Then from Matrix Christofell-Darboux formula

$$\overleftarrow{\phi}_{n,m}(z,w) \overline{\overleftarrow{\phi}_{n,m}(z_1,w_1)} - \phi_{n,m}(z,w) \overline{\phi_{n,m}(z_1,w_1)}$$

$$= (1 - z\bar{z}_1) \tilde{\Phi}_{n-1,m}(z,w)^T \tilde{\Phi}_{n-1,m}^*(z_1,w_1)^T \qquad .$$

$$+ (1 - w\bar{w}_1) \overleftarrow{\Phi}_{n,m-1}(z,w) \overleftarrow{\Phi}_{n,m-1}^*(z_1,w_1)^T$$

$K_{n,m} = 0$ is a geometric condition which can be translated into a low rank condition on certain submatrices of $C_{n,m}$. Thus if $C_{n,m}$ is positive definite and satisfies certain low rank conditions then there exists a causal solution to the autoregressive filter problem discusses above.

As in the one variable case this gives stability and spectral matching of $\overleftarrow{\phi}_{n,m}(z, w)$

If $\overleftarrow{\phi}_{n,m}$ is stable then

$$d\mu(\theta, \phi) = \frac{1}{4\pi^2} \frac{d\theta d\phi}{|\phi_{n,m}(\theta, \phi)|^2}$$

is a two variable analog of a Bernstein-Szego measure.

A rational function R of n complex variables $z_1, ..z_n$ is the ratio of two polynomials of n complex variables i.e $R = \frac{P}{Q}$. Note we can eliminates common factors of P and Q however their zero sets may still intersect. For examples

$$R(z_1, z_2) = z_1/z_2.$$

The degree of a polynomial P is the maximum of the total degrees of the monomials appearing in P with nonzero coefficients.

A rational inner function R of n complex variables is a rational function such that $R \in H^\infty(U^n)$ such that $|R(z)| = 1$ a.e $z \in T^n$

Rudin has shown that every rational inner function of n comples variables R is of the form

$$R(z) = \frac{M(z)\bar{Q}(1/z)}{Q(z)},$$

where Q is a polynomials in z that is nonzero in U^n and M is a monomial of sufficiently high degree so that the above numerator is a polynomial.

Caratheodory Theorem

Theorem Every $f \in H^\infty(U^n)$ with $|f| \leq 1$ is a limit (uniformly on compact subsets of U^n) of a sequence of rational inner functions which are continuous on $\bar{U}^n$.

Christoffel-Darboux again gives

$$
\begin{aligned}
&1 - f(z,w)\overline{f(z_1,w_1)} \\
&= (1 - z\bar{z}_1)\sum_{j=0}^{n} f_j(z,w)\overline{f_j(z_1,w_1)} \\
&+ (1 - w\bar{w}_1)\sum_{k=0}^{m} g_k(z,w)\overline{g_k(z_1,w_1)},
\end{aligned}
$$

where f is a rational inner function on D^2.

This gives new proof of Ando's see paper of Cole-Wermer

Theorem

Let T_1 and T_2 be two commuting strict contractions on a Hilbert space H. If P is a polynomial on D^2 then

$$||P(T_1,T_2)|| \leq \sup_{D^2} |P|$$

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