In 1920, Hardy proved what today is one of the classical inequalities on integration:

$$\left( \int_0^\infty \left( 1 + \int_0^t f(x) \, dx \right)^p \, dt \right)^{1/p} \leq \frac{p}{p-1} \left( \int_0^\infty (f(x))^p \, dx \right)^{1/p}$$

for $1 < p < \infty$ and $f$ any positive measurable real function. This inequality has been studied by a lot of mathematicians producing a great variety of Hardy type inequalities which apply in different areas of mathematics.

Considering the average of functions as an operator $S$, i.e.

$$Sf(x) = \frac{1}{\theta(x)} \int_0^x f(s) \, ds, \quad x > 0,$n

for $f \in L^1_{\theta}(0, \infty)$, the Hardy’s inequality establishes that $S : L^p(0, \infty) \to L^p(0, \infty)$ is well defined and continuous. This $S$ is called the Hardy operator.

- Is there any space $Y$ larger than $L^p(0, \infty)$ such that $S : Y \to L^p(0, \infty)$?
- Which is the largest space $Y$ such that $S : Y \to L^p(0, \infty)$?

The space $[S, X]$ is the largest r.i. B.f.s. $X$ such that $S : L^p(0, \infty) \to X$.

**Theorem.** Suppose that $\varphi$ satisfies

- $\varphi(0^+) = 0$.
- $\varphi(t) = 0$ for all $t > 0$.
- $\varphi$ is non-decreasing on $[0, \infty)$.

Then,

$$\left\{ \frac{1}{\varphi(t)} \int_0^t f(s) \, ds : f \in L^p(0, \infty) \right\}$$

is a B.f.s. with the norm $\|f\|_{[S, X]} = \left( \int_0^\infty f^p(x) \, dx \right)^{1/p}$.

Let $X$ be a Banach function space (B.f.s.), i.e. a Banach space of measurable real functions on $[0, \infty)$, satisfying $|f| \leq \varphi(|g|)$ for all $f \in X$ and $\|f\|_X \leq \|g\|_X$.

- Is there any B.f.s. $Y$ such that $S : Y \to X$?
- Which is the largest B.f.s. $Y$ such that $S : Y \to X$?

Note that if $S : Y \to X$ is well defined is automatically continuous, since it is a positive operator between Banach lattices.

**Particular cases of $[S, X]$**

- $[S, L^1(0, \infty)] = (0)$.
- $[S, L^\infty(0, \infty)] = L^1(0, \infty)$, for any r.i. B.f.s.

**R.I. optimal domain for $S$**

**R.I. optimal domain for $S$**

The space $[S, X]$ may not be r.i., nor a natural question is:

Which is the largest r.i. B.f.s. contained in $[S, X]$?

The space $[S, X]_{\text{r.i.}} = \{ f : [0, \infty) \to \mathbb{R} : \text{measurable} : \int_0^\infty (f^p(t))^{1/p} \, dt < \infty \}$, for any r.i. B.f.s. $X$.

**Particular cases of $[S, X]_{\text{r.i.}}$**

- $[S, L^1(0, \infty)]_{\text{r.i.}} = (0)$.
- $[S, L^\infty(0, \infty)]_{\text{r.i.}} = L^1(0, \infty)$.

**When does $X$ coincide with $[S, X]_{\text{r.i.}}$?**

For every r.i. B.f.s. $X$, since $f^p \leq Sf^p$, it follows that $[S, X]_{\text{r.i.}} \subset X$. If $X$ also satisfies $S : X \to X$, then $[S, X]_{\text{r.i.}} = X$. So, since $[S, X]_{\text{r.i.}}$ is the largest r.i. B.f.s. contained in $[S, X]$, we have that $[S, X]_{\text{r.i.}} = X$.

Conversely, suppose $[S, X]_{\text{r.i.}} = X$. Then $X$ is obviously r.i. and, since $|f| \leq S|f| \leq Sf^p$, it follows that $S : X \to X$. Therefore, the following proposition holds.

**Proposition.** $X$ is r.i. and $S : X \to X$ if and only if $X = [S, X]_{\text{r.i.}}$.

In particular,

$$[S, L^p(0, \infty)]_{\text{r.i.}} = L^p(0, \infty)$$

for all $1 < p < \infty$.