A family of quasi-birth-and-death processes coming from the theory of matrix valued orthogonal polynomials* 

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Birth and death processes

The birth-and-death processes are a very special kind of Markov chain on the space of non-negative integers. At each discrete unit of time a transition is allowed from state i to state j with probability p_{ij} and we put p_{ii} = 0 if i = j. The one-step transition probability matrix is given by

\[ P = \begin{pmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

The diagram of the process looks as follows:

[Diagram of a birth-and-death process]

The problem here is to obtain an expression of the so called n-step transition probability matrix, giving the probability of going between any two states in n steps. S. Karlin and J. McGregor, [3], obtained a neat representation formula for this quantity of interest. Introducing the sequence of polynomials \( \{P_n(x)\} \) by the conditions \( P_n(0) = 0, P_n(1) = 1 \), using the notation \( \alpha = (\alpha_1, \alpha_2, \cdots) \) and insisting on the recursion relation \( P_n = \chi \alpha \), it is possible to prove the existence of a unique measure \( d\nu(x) \) supported in \([-1,1]\) that makes the polynomials \( \{P_n(x)\} \), orthogonal.

Another probabilistic object of interest is so called the invariant (stationary) distribution, i.e.

\[ \pi_n = \alpha_{n+1} \cdot \cdots \cdot \alpha_1 \cdot \frac{1}{2^n} p_{00}(x) \]

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Quasi-birth-and-death processes

The quasi-birth-and-death processes are a natural extension of the birth-and-death processes where the one-step transition probability matrix is a block diagonal matrix (each block of size \( x \times d \)).

\[ P = \begin{pmatrix} B_0 & 0 & 0 & \cdots \\ 0 & B_1 & 0 & \cdots \\ 0 & 0 & B_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

We now have two dimensional Markov chain with discrete time. The state space consists of the pair of integers \( (i, j), \ i, j \in \{0,1,2,\ldots\} \). The first component is usually called the level and the second one the phase. This indicates that in one unit of time a transition can change the phase without changing the level, or can change the level (and possibly the phase) to either of the adjacent levels. The probability of going in one step from state \( (i, j) \) to state \( (i', j') \) is given by the \( (i, j) \) element of the block \( P_{ij} \).

As before, introducing the matrices polynomials \( \{Q_n(x)\} \) by the conditions \( Q_n(0) = 0, Q_n(1) = I \), using the notation \( \alpha = (\alpha_1, \alpha_2, \cdots) \) and insisting on the recursion relation \( P_n = \chi \alpha \), it is possible to prove (under certain technical conditions on the coefficients \( \alpha_0, \alpha_n, \alpha_0 \)) the existence of a unique weight measure \( d\mu(x) \) supported in \([-1,1]\) that makes the polynomials \( \{Q_n(x)\} \), orthogonal.

Again, one gets the Karlin-McGregor representation formula [see (1, 2)]

\[ \mu = \left( 1 + x_{1}^{2} \right) \left( 1 + x_{2}^{2} \right) \cdots \left( 1 + x_{n}^{2} \right) \left( 1 + x_{n+1}^{2} \right) \cdots \left( 1 + x_{d}^{2} \right) \]

Note, the problem of computing an invariant distribution row vector with non-negative entries \( \pi_j \)

\[ \pi = (\pi_0, \pi_1, \cdots) = (\pi_0, \pi_1, \cdots, \pi_n, \pi_{n+1}, \cdots) \]

such that \( \pi \pi = \mu \) leads to a complicated system of equations. A formula that relates the entries \( \pi_j \)

References


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